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VII. *Ellipsoidal Functions and their Application to some Wave Problems.*

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Introduction.

The analysis of wave motion connected with the circular disc and circular aperture is of such great analytical and physical interest, that no excuse is required for a somewhat extended treatment.

The theory of wave motion has been very fully developed when the surfaces bounding the medium in which the motion takes place are spheres and circular cones, or cylinders and planes. The functions involved in the analysis are of the hypergeometric type.

Next in order of importance is the theory required for the treatment of problems in which the surfaces bounding the medium are :—

A. Ellipsoids of revolution and hyperboloids of revolution of one and two sheets ;
B. Elliptic and hyperbolic cylinders.

The functions involved in case A are functions associated with the ellipsoid of revolution, or spheroid.

The functions involved in case B are commonly called Mathieu functions.

The Mathieu functions and the functions associated with the ellipsoid of revolution are of considerable importance but, unfortunately, of great complexity. The latter functions are, however, not too complicated to be of considerable use and it is, accordingly, proposed to consider them in the present paper, especially in connection with the ellipsoids of revolution and the hyperboloids of revolution of *one* sheet. The functions have application to a number of important problems, among which the following may be mentioned :—

(1) The oscillations of a gas in an ellipsoidal envelope.
(2) The passage of waves through a circular aperture in a plane screen.
(3) The diffraction of waves by a circular disc.
(4) The oscillations of a circular disc in an unlimited medium.

There appear to be a number of ways in which these problems may be attacked, and the development of many of these ways may be of interest to the mathematician. But such a development would obviously extend the paper quite unreasonably. It

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has therefore been necessary to keep in view the physical aspects of the problems considered. In the first place solutions of a differential equation are obtained by means of an integral equation suggested by WHITTAKER, and some important properties of the functions involved are found.

In the second place some transformations of the first obtained differential equations are considered. It is a matter of difficulty to decide upon those transformations which are likely to be the most useful both from the point of view of the mathematician and that of the physicist. Of the three considered two were discarded for reasons given in the text, but none the less it is thought proper to mention them as they may provide a useful line of attack in other problems.

The elementary solutions of the wave equation may be written in the form

$$\{AE_n(\mu) + BE'_n(\mu)\}\{CH_n(\rho) + DH'_n(\rho)\},$$

in which A, B, C, D, are constants of integration. Here $AE_n + BE'_n$ is a sum of functions $\alpha_s(AP_s + BQ_s)$, in which P is LEGENDRE's function, of the first kind and Q LEGENDRE's function of the second kind.

Also $CH_n + DH'_n$ is a sum of functions $\rho^{-\frac{1}{2}}\beta_s(CJ_{s+\frac{1}{2}} + DJ_{-s-\frac{1}{2}})$, in which J is BESSEL's function of fractional order.

In the above, n and s are integers.

The general solution of the wave equation is thus obtained, but from the point of view of the applications it will be shown that Q_s and $J_{-s-\frac{1}{2}}$ must be discarded. The solutions E'_n and H'_n must therefore be obtained in another way.

For our purpose H'_r is the most important, as it is necessary for the divergent wave, and the method used for obtaining it is a well-known one in the theory of differential equations.

No alternative method of dealing with the divergent wave appears suitable.

Unfortunately the method employed necessitates a great amount of algebraical development, the coefficients being somewhat cumbersome. Nevertheless the development of the coefficients is analytically of considerable interest, and the task of obtaining some results of physical interest is not unduly laborious.

Although the methods of solution involving the theory of integral equations and the theory of continued approximation are shown to lead to some useful results, the most generally serviceable method is that of series, especially in the case of the second solution which is required for the divergent wave. As the paper is somewhat long the discussion of the results obtained in the cases of the disc and the aperture has been considerably curtailed, as also that of the oscillations within a spheroidal envelope.

The analysis is, for the most part, very different from that encountered in the problem of the sphere. But, as in the latter case, RAYLEIGH's remark, that the harmonic analysis seems to lose its power when the waves are very small in comparison with the dimensions of the bodies, still holds good.

NICHOLSON,* in the development of a solution of a special problem, follows a method suggested by that which STOKES employed in the case of the sphere. The required solution is, however, not that of the wave equation, and is therefore not applicable directly in the present paper.

Of great importance in the theory is the ratio of the wave-length to the diameter of the focal circle, which is common to the family of ellipsoids and hyperboloids of one sheet.

When this ratio is greater than five, that is to say, when the wave-length is more than five times the diameter of the focal circle, the analysis required to obtain tolerable accuracy is little more complicated than that required for the sphere.

For the special cases of the passage of waves through a small aperture and the diffraction of waves by a small ellipsoid, a well-known method, devised by RAYLEIGH, is available.

When the wave-length is less than one-half of the diameter of the focal circle the analysis required to obtain reasonable accuracy becomes unwieldy.

For certain problems in this latter case, other approximate methods become available with some approach to accuracy, but they are very restricted. But between the above-mentioned extremes there lie certain problems of great interest, to which the analysis developed in the present paper can be successfully applied without very much labour.

Both for this reason and for its own intrinsic interest it is hoped that the space devoted to analytical development is fully justified.

An index of symbols, and the number of equation from which the definition of each symbol can be most easily found, is subjoined :—

Symbol.	*Description.*	*Equation.*
Φ	Wave function	1
r, z, ω	Cylindrical co-ordinates	1
c	Wave velocity	1
t	Time	1
a	Radius of focal circle	2
α, θ	Parameters of transformation	2
μ	$= \cos\theta$	3
β	$= \sinh\alpha$	3
λ	Wave-length	3
κ_1	$= 2\pi/\lambda$	3
κ	$= \kappa_1 a$	3*a*
$E_n(\mu)$	Elementary solution	4
a_n	Function of κ	4

* "The Symmetrical Vibrations of Conducting Surfaces of Revolution." 'Phil. Mag.,' vol. 11, p. 703 (1906).

Symbol.	*Description.*	*Equation.*
$H_n(\kappa\beta)$	Elementary solution	5
ρ	$= \kappa\beta$	5
R	$= (r^2 + z^2)^{\frac{1}{2}}$	6
λ_n	Characteristic number	7
ϕ	Variable of integral equation	7
α'_n	Numerical constants	10
p_n	Functions of LEGENDRE's polynomials	11
$\beta_{n,m}$	Coefficient of P_m in expansion of E_n	12
e_n	Coefficient defined by integral	17
∇	Operator of left side of equations	20, 24, 27, 32
D	Operator of right side of equations	20, 24, 27, 32
$L_s M_s N_s A_s \alpha_s$	Coefficients associated with the operators ∇ and D.	—
G_n	$= e^{-\kappa\mu} E_n$	24
R_n	$= \rho^{\frac{1}{2}} H_n$	27
F_n	$= J_{n+\frac{1}{2}}$ (BESSEL's function)	27
S_n	$= \sigma^{\frac{1}{2}} H_n$	32
σ^2	$= \rho^2 + \kappa^2$	32
$\gamma_{n,m}$	Coefficient of F_m in expansion of R_n	37
b_n	$= -M_n$	40, 41
$L'_s M'_s N'_s A'_s$	Coefficients associated with $L_s M_s N_s A_s$	47–51
Δ_n	Infinite determinant	48
Δ'_n	Finite determinant	49
u, v, w	Variables of divergent wave	56
T_n	$= \rho^{\frac{1}{2}} w$	61
$B_{n,m}$	Coefficient of F_m in the expansion of the last term on the right-hand side of (61)	64
I_s	Part solution of (61)	66
$c_{s,m}$	Coefficient of F_m in $\alpha_s I_s$	66
$\varepsilon_{n,m}$	Coefficient of F_m in T_n	69
H'_n	Second solution of (5)	74
A_n	Coefficient of H'_n in expression for divergent wave	75
D_n	Denominator of A_n	75
ψ_n	$= H_n + A_n H'_n$	95

The Wave Equation.

The family of surfaces constituting the Planetary Ellipsoids or Spheroids which possess a common focal circle is bounded at one end by a circular disc whose perimeter is the focal circle. At the other end the surfaces tend to become more and more spherical in form.

The orthogonal family of Hyperboloids of one sheet is bounded at one end by an infinite plane pierced by a circular aperture whose perimeter is the focal circle. At the other end the boundary of the family is the common axis of revolution or symmetry.

Both families are symmetrical about this common axis, which will be taken as the axis of z.

Let Φ be the wave function.

The plane of the common focal circle being taken as the plane of xy, and cylindrical co-ordinates being adopted so that

$$x = r\cos\omega$$

$$y = r\sin\omega,$$

the wave equation is expressed in the form

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\omega^2} + \frac{\partial^2\Phi}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 0, \quad \ldots\ldots\ldots \quad (1)$$

where c is the wave velocity.

In problems concerned with the planetary ellipsoids equation (1) is transformed by the substitution

$$z + ir = a\sinh(\alpha + i\theta). \quad \ldots\ldots\ldots\ldots \quad (2)$$

This transformation gives

$$z = a\sinh\alpha\cos\theta$$

$$r = a\cosh\alpha\sin\theta.$$

The circle, $z = 0$, $r = a$, is the focal circle. The surfaces, $\alpha =$ const., constitute the family of planetary ellipsoids. The surfaces, $\theta =$ const., constitute the family of confocal hyperboloids.

For problems in which one of the family of ellipsoids bounds space internally the function θ varies from $\theta = 0$ along the positive direction of the z axis to $\theta = \pi$ along the negative direction.

The function α is everywhere positive.

This is also the case if space be bounded internally and externally by ellipsoidal surfaces.

For problems in which space is bounded by one of the family of hyperboloids the only case of importance is that in which the space under consideration contains the common axis of symmetry. In this case the function θ is zero along the whole of the axis of z. The function α changes sign upon crossing the plane $\alpha = 0$.

So far as the dependence of Φ upon time is concerned it will be assumed that the motion can be resolved into its harmonic constituents, so that it will be sufficient to make

$$\Phi \propto e^{i\kappa_1 ct},$$

where the wave-length corresponding to this harmonic constituent is

$$\lambda = 2\pi/\kappa_1.$$

Write now

$$\sinh\alpha = \beta,$$

and

$$\cos\theta = \mu,$$

so that

$$z = a\beta\mu,$$

and

$$r = a(\beta^2 + 1)^{\frac{1}{2}}(1 - \mu^2)^{\frac{1}{2}}.$$

By means of the general theory of curvilinear co-ordinates elements of length are expressed by

$$\delta s_\mu = a\left\{\frac{\beta^2 + \mu^2}{1 - \mu^2}\right\}^{\frac{1}{2}}\delta\mu$$

$$\delta s_\beta = a\left\{\frac{\beta^2 + \mu^2}{\beta^2 + 1}\right\}^{\frac{1}{2}}\delta\beta$$

$$\delta s_\omega = a(\beta^2 + 1)^{\frac{1}{2}}(1 - \mu^2)^{\frac{1}{2}}\delta\omega.$$

The wave equation (1) in these co-ordinates is

$$\frac{\partial}{\partial\mu}\left\{(1-\mu^2)\frac{\partial\Phi}{\partial\mu}\right\} + \frac{\partial}{\partial\beta}\left\{(\beta^2+1)\frac{\partial\Phi}{\partial\beta}\right\} + \frac{\beta^2+\mu^2}{(1-\mu^2)(\beta^2+1)}\cdot\frac{\partial^2\Phi}{\partial\omega^2} + \kappa_1^2 a^2(\beta^2+\mu^2)\Phi = 0. \quad (3)$$

So far as its dependence upon ω is concerned

$$\Phi \propto \cos(m\omega + \varepsilon),$$

where m and ε are constants.

If, then, we put

$$\kappa_1 a = \kappa$$

equation (3) reduces to

$$\frac{\partial}{\partial\mu}\left\{(1-\mu^2)\frac{\partial\Phi_1}{\partial\mu}\right\} + \frac{\partial}{\partial\beta}\left\{(\beta^2+1)\frac{\partial\Phi_1}{\partial\beta}\right\} + \left\{\kappa^2\mu^2 - \frac{m^2\mu^2}{1-\mu^2}\right\}\Phi_1 + \left\{\kappa^2\beta^2 - \frac{m^2\beta^2}{\beta^2+1}\right\}\Phi_1 = 0, \quad (3a)$$

where

$$\Phi = \Phi_1\cos(m\omega + \varepsilon).$$

Let

$$\Phi_1 = \text{E}\cdot\text{H},$$

where E is a function of μ only and H is a function of β only.

The functions E and H are the elementary solutions of the equations

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dE}{d\mu}\right\}+\left\{\gamma-\kappa^2(1-\mu^2)-\frac{m^2}{1-\mu^2}\right\}E=0,$$

and

$$\frac{d}{d\beta}\left\{(\beta^2+1)\frac{dH}{d\beta}\right\}-\left\{\gamma-\kappa^2(\beta^2+1)-\frac{m^2}{\beta^2+1}\right\}H=0,$$

corresponding to γ.

It may be noted that, if $i\beta'$ be substituted for β, the second equation, considered as an equation in β', is of exactly the same form as the first equation in μ.

As we shall consider in this paper only solutions of symmetry about the axis of revolution, m may be made zero. Hence, omitting a constant factor, we may put, in cases of symmetry,

$$\Phi = \Phi_1.$$

Write now

$$\gamma = n(n+1) + a_n$$

where n is a positive integer.

Let E_n and H_n be the elementary solutions corresponding to this value of γ.

Writing

$$\rho = \kappa\beta = \kappa_1 a \sinh\alpha,$$

the equations become

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dE_n}{d\mu}\right\}+\{n(n+1)+a_n-\kappa^2(1-\mu^2)\}E_n=0, \quad . \quad . \quad . \quad (4)$$

and

$$\frac{d}{d\rho}\left\{(\rho^2+\kappa^2)\frac{dH_m}{d\rho}\right\}-\{n(n+1)+a_n-(\rho^2+\kappa^2)\}H_n=0. \quad . \quad . \quad . \quad . \quad (5)$$

Suppose that $a \longrightarrow 0$ and $e^\alpha \longrightarrow \infty$ in such a way that ae^α remains finite.

Then

$$\rho \longrightarrow \kappa_1 R,$$

where

$$R^2 = r^2 + z^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If therefore a, and consequently κ as it occurs *explicitly*, is made zero in (4) and (5), those equations must reduce to the well-known equations in spherical polar co-ordinates. It will be seen presently that a_n must vanish when κ vanishes and n is an integer.

The equations (4) and (5) may be taken as the standard equations. The substitution $i\beta'$ for β is found to be unsuitable in physical applications, unless $\kappa = 0$.

The Functions E_n and the Integral Equation.

In physical problems space is limited by the boundaries. Throughout this space the functions E_n must be finite, single valued, and continuous. If an ellipsoid is a boundary they must be continuous in the sense that, when θ is increased by 2π, they regain their original values. This condition is sufficient to determine the value of a_n in equation (4) which value is not given *à priori*.

If a hyperboloid is a boundary the condition may not be obvious. But it will be shown that in all cases it is impossible to satisfy equation (5) by a permissible function unless a_n has the same value.

Functions of μ ($= \cos\theta$) are called even functions, so that E_n is analogous to the MATHIEU function ce_n.

Even functions only will be considered in this paper, since they alone are required for the most important problems.

We shall first obtain the functions E_n independently and develop some of their properties. We shall then consider the functions E_n and H_n concurrently.

It will be shown in the first place, following WHITTAKER's* treatment of MATHIEU's equations, that the function E_n satisfies the integral equation

$$E_n(\mu) = \lambda_n \int_{-1}^{1} e^{\kappa\mu\phi}\, E_n(\phi)\, d\phi. \qquad (7)$$

The theory of these equations is now an extensive subject. The integral equation in question is one with a symmetric nucleus and a characteristic number λ_n. To the characteristic number, there corresponds the characteristic function E_n. Two such functions, possessing different characteristic numbers, are mutually orthogonal, a property which is of great utility. The range for which they are orthogonal is $(-1, 1)$.

Consider the equation

$$(a\mu^2 + b)\frac{d^2E}{d\mu^2} + 2a\mu\frac{dE}{d\mu} + (\kappa^2 b\mu^2 + c)\,E = 0, \qquad (8)$$

where $\mu = \cos\theta$.

Let

$$E(\mu) \propto \int e^{\kappa\mu\phi}\, E(\phi)\, d\phi.$$

Substituting this expression in (8) it is found that

$$\int \{\kappa^2\mu^2(a\phi^2 + b) + \kappa\mu(2a\phi) + (\kappa^2 b\phi^2 + c)\}\, e^{\kappa\mu\phi}\, E(\phi)\, d\phi = 0.$$

Upon integrating this equation by parts, it becomes

$$\left| \{\kappa\mu E(\phi) - E'(\phi)\}(a\phi^2 + b)\, e^{\kappa\mu\phi} \right|$$
$$+ \int \{(a\phi^2 + b)\, E''(\phi) + 2a\phi E'(\phi) + (\kappa^2 b\phi^2 + c)\}\, e^{\kappa\mu\phi}\, E(\phi)\, d\phi = 0, \qquad (9)$$

where the dashes denote differentiation with respect to ϕ.

* WHITTAKER and WATSON, "Modern Analysis," p. 400.

Since $E(\phi)$ is the same function of ϕ as $E(\mu)$ is of μ, it satisfies the same differential equation (8). Hence that part of equation (9) under the integral sign vanishes.

The first part of equation (9) has the factor $a\phi^2 + b$, and therefore this part of equation (9) vanishes when the factor is zero. The factor determines the limits of the integral.

Hence, upon comparing (8) with (4),

$$E_n(\mu) = \lambda_n \int_{-1}^{1} e^{\kappa\mu\phi} E_n(\phi)\, d\phi,$$

an equation which has a continuous solution only for a determinate value of λ_n as a function of κ.

In accordance with the theory of integral equations $1/\lambda_n$ may be expanded in a series of ascending integral powers of κ. Hence we may write, as will presently be manifest,

$$1/\lambda_n = 2\,(\ldots + \alpha'_{n-2}\,\kappa^{n-2} + \alpha'_n\,\kappa^n + \alpha'_{n+2}\,\kappa^{n+2} + \ldots). \quad . \quad . \quad . \quad . \quad (10)$$

From an inspection of the differential equation (4) it may be assumed that $E_n(\mu)$ can be expanded in a series

$$P_n(\mu) + \kappa^2 p_2(\mu) + \kappa^4 p_4(\mu) + \ldots,$$

where P_n is LEGENDRE's polynomial of order n. Further the p's contain only LEGENDRE polynomials of even order if n be even, and only of odd order if n be odd.

Expanding the exponential in (6) and inserting the assumed series for $1/\lambda_n$ and E_n, it is clear that, when n is even, all the terms of odd degree in ϕ in the exponential vanish upon integration between the limits, and, when n is odd, all the terms of even degree in ϕ vanish.

It is also evident that, since

$$\int_{-1}^{1} P_n(\phi)\,.\,\phi^m\, d\phi$$

vanishes when $m < n$,

$$\alpha'_{n-2},\ \alpha'_{n-4},\ \text{etc., all vanish.}$$

The integral equation (6) may therefore be written in the form,

$$\tfrac{1}{2}\int_{-1}^{1}\left\{\ldots + \frac{\kappa^{n-2}\mu^{n-2}}{(n-2)!}\,\phi^{n-2} + \frac{\kappa^n\mu^n}{n\,!}\,\phi^n + \ldots\right\}\{P_n(\phi) + \kappa^2 p_2(\phi) + \ldots\}\, d\phi$$
$$= \{\alpha'_n\kappa^n + \alpha'_{n+2}\kappa^{n+2} + \ldots\}\{P_n(\mu) + \kappa^2 p_2(\mu) + \ldots\}. \quad . \quad . \quad (11)$$

Upon equating coefficients of κ^n in (11) it is found that that equation is satisfied if p_2 contains no LEGENDRE polynomial of order less than $n-2$; if p_4 contains none of order less than $n-4$; and so on. Upon equating coefficients of κ^{n+2} it is found that p_2 cannot contain a LEGENDRE polynomial of order higher than $n+2$; p_4 cannot contain one of order higher than $n+4$; and so on.

Further, none of the p's contain P_n.

For the expansion of E_n we shall write,

$$E_n = \ldots + \beta_{n,\,n-2}\,P_{n-2} + P_n + \beta_{n,\,n+2}\,P_{n+2} + \ldots . \qquad (12)$$

Let 2_{n-2} be the coefficient of κ^2 in $\beta_{n,\,n-2}$;
Let 4_{n-2} be the coefficient of κ^4 in $\beta_{n,\,n-2}$;
Let 4_{n-4} be the coefficient of κ^4 in $\beta_{n,\,n-4}$;
with a corresponding notation throughout. The coefficients 2_{n-2}, 4_{n-2}, etc., are, therefore, functions of n only.

For the evaluation of these coefficients certain formulæ, connected with the LEGENDRE polynomials, are required.

Writing

$$B_m = \frac{(2m)\,!}{2^m\,(m\,!)^2},$$

it is not difficult to show that

$$P_m(\phi) = B_m\,\phi^m - \tfrac{1}{2}(m-1)\,B_{m-1}\,\phi^{m-2} + \tfrac{1}{8}(m-2)(m-3)\,B_{m-2}\,\phi^{m-4} - \ldots .$$

The integrals occurring in (11) can be simplified as follows :—

Let

$$I_m = \frac{1}{m\,!}\cdot\tfrac{1}{2}\int_{-1}^{1}\phi^m P_m(\phi)\,d\phi,$$

$$= \frac{2^m m\,!}{(2m+1)\,!}.$$

Then

$$\frac{1}{m\,!}\cdot\tfrac{1}{2}\int_{-1}^{1}\phi^m P_{m-2}(\phi)\,d\phi = \tfrac{1}{2}(2m+1)\,I_m,$$

$$\frac{1}{m\,!}\cdot\tfrac{1}{2}\int_{-1}^{1}\phi^m P_{m-4}(\phi)\,d\phi = \tfrac{1}{8}(2m-1)(2m+1)\,I_m,$$

and so on.

Equation (11) must now be substituted in (12), and the resulting equation must be an identity so far as both κ and μ are concerned.

Equate in the first place coefficients of corresponding powers of κ, and then coefficients of corresponding powers of μ, and after some reduction we obtain

$$2_{n-2} = -\frac{n(n-1)}{2(2n+1)(2n-1)^2},$$

$$2_{n+2} = \frac{(n+1)(n+2)}{2(2n+1)(2n+3)^2},$$

since

$$\alpha'_n B_n = I_n.$$

Further,

$$\left.\begin{aligned}
\alpha'_{n+2} &= \tfrac{1}{2}(2n+1)(2_{n-2}+2_{n+2}),\\
4_{n-4} &= \frac{n(n-1)(n-2)(n-3)}{8(2n+1)(2n-1)^2(2n-3)^2(2n-5)},\\
4_{n+4} &= \frac{(n+1)(n+2)(n+3)(n+4)}{8(2n+1)(2n+3)^2(2n+5)^2(2n+7)},\\
4_{n-2} &= \frac{n(n-1)}{(2n+1)(2n-1)^2}\left[-\tfrac{1}{2}\alpha'_{n+2}+\frac{2_{n-2}}{2n-3}+\tfrac{1}{8}(2n-1)2_{n+2}\right.\\
&\qquad\left.-\frac{(n-2)(n-3)}{16(2n-3)(2n-5)}\right],\\
4_{n+2} &= \frac{(n+1)(n+2)}{(2n+1)(2n+3)^2}\left[-\tfrac{1}{2}\alpha'_{n+2}+\frac{2_{n+2}}{2n+5}+\tfrac{1}{8}(2n+3)2_{n-2}\right.\\
&\qquad\left.+\frac{(n+3)(n+4)}{16(2n+5)(2n+7)}\right].
\end{aligned}\right\} \quad \ldots\ldots (13)$$

The numerical values of the coefficients decrease rapidly as n increases, but the actual general values become very complicated for higher powers of κ. If, however, κ be not too great it is unnecessary to consider powers of κ higher than the fourth.

For reference we may write down the solutions $E_n(\mu)$, when $n = 0, 1, 2, 3$, as far as the fourth power of κ.

$$\left.\begin{aligned}
E_0(\mu) &= P_0 + \left\{\left(\frac{1}{3}\kappa\right)^2 + \frac{2}{7}\left(\frac{1}{3}\kappa\right)^4\right\}P_2 + \frac{3\cdot 9}{7\cdot 25}\left(\frac{1}{3}\kappa\right)^4 P_4\\
E_1(\mu) &= P_1 + \left\{\left(\frac{1}{5}\kappa\right)^2 - \frac{2}{9}\left(\frac{1}{5}\kappa\right)^4\right\}P_3 + \frac{5\cdot 25}{9\cdot 49}\left(\frac{1}{5}\kappa\right)^4 P_5\\
E_2(\mu) &= -\frac{1}{5}\left\{\left(\frac{1}{3}\kappa\right)^2 + \frac{2}{7}\left(\frac{1}{3}\kappa\right)^4\right\}P_0 + P_2\\
&\qquad + \left\{\frac{6}{5}\left(\frac{1}{7}\kappa\right)^2 - \frac{4}{55}\left(\frac{1}{7}\kappa\right)^4\right\}P_4 + \frac{49}{11\cdot 9}\left(\frac{1}{7}\kappa\right)^4 P_6\\
E_3(\mu) &= -\frac{3}{7}\left\{\left(\frac{1}{5}\kappa\right)^2 - \frac{2}{9}\left(\frac{1}{5}\kappa\right)^4\right\}P_1 + P_3\\
&\qquad + \left\{\frac{10}{7}\left(\frac{1}{9}\kappa\right)^2 - \frac{4}{91}\left(\frac{1}{9}\kappa\right)^4\right\}P_5 + \frac{15\cdot 81}{13\cdot 121}\left(\frac{1}{9}\kappa\right)^4 P_7.
\end{aligned}\right\} \quad \ldots\ldots (14)$$

The variable μ is understood in the LEGENDRE polynomials.

The similarity between the coefficient of P_2 in E_0 and that of P_0 in E_2, as also between that of P_3 in E_1 and that of P_1 in E_3, is due to the orthogonal property of the functions.

This brief survey of the integral equation is sufficient so far as the values of the coefficients are concerned. These will be confirmed and extended later by an entirely different method, which it is convenient to adopt when developing a converging solution for the functions in connection with the divergent wave.

The latter method is also more convenient for determining the value of a_n when κ tends to become large.

The nature of the convergence of the coefficients is also more easily investigated thereby.

The Orthogonal Property.

A very valuable property of the functions which satisfy an integral equation with a symmetric nucleus is the orthogonal property. It provides a simple method of expressing an arbitrary function throughout a given interval in a series of the functions.

If $E_n(\mu)$ and $E_m(\mu)$ be two ellipsoidal functions, then, if $n \neq m$,

$$\int_{-1}^{1} E_n(\mu) E_m(\mu) \, d\mu = 0. \quad \text{(15)}$$

It can be shown that the expansion of an arbitrary function in the range $(-1, 1)$ is thus determinate.

The expression for a plane wave takes a remarkably simple form, very similar to RAYLEIGH's well-known formula in spherical polar co-ordinates. RAYLEIGH's formula is

$$e^{i\rho\mu} = \sum_{n=0}^{n=\infty} (2n+1) \, i^n \, (\tfrac{1}{2}\pi)^{\frac{1}{2}} \frac{1}{\rho^{\frac{1}{2}}} J_{n+\frac{1}{2}}(\rho) \, . \, P_n(\mu). \quad \text{(16)}$$

It is convenient to write

$$\begin{aligned} e_n &= \int_{-1}^{1} E_n^2(\mu) \, d\mu \\ &= \int_{-1}^{1} \{\dots + \beta^2_{n,n-2} P^2_{n-2} + P^2_n + \beta^2_{n,n+2} P^2_{n+2} + \dots\} \, d\mu \\ &= \frac{2}{2n+1} \left\{\dots + \frac{2n+1}{2n-3} \beta^2_{n,n-2} + 1 + \frac{2n+1}{2n+5} \beta^2_{n,n+2} + \dots\right\} \quad \text{(17)} \end{aligned}$$

Let it be required to expand the function $f(\mu)$ in a series of the ellipsoidal functions. Assume that

$$f(\mu) = a_0 E_0 + a_1 E_1 + \dots + a_n E_n + \dots .$$

Then

$$a_n e_n = \int_{-1}^{1} E_n(\mu) f(\mu) \, d\mu.$$

To find the expansion of μ^m, we have

$$\begin{aligned} a_n e_n &= \int_{-1}^{1} \mu^m \{\dots + \beta_{n,n-2} P_{n-2} + P_n + \beta_{n,n+2} P_{n+2} + \dots\} \, d\mu \\ &= \dots + \beta_{n,n-2} \, h_{m,n-2} + h_{m,n} + \beta_{n,n+2} \, h_{m,n+2} + \dots , \end{aligned}$$

where

$$h_{m,n} = \int_{-1}^{1} \mu^m \mathrm{P}_n(\mu)\, d\mu$$

$$= \frac{2^{n+1}\, m\,!\, (\tfrac{1}{2}n + \tfrac{1}{2}m)\,!}{(\tfrac{1}{2}m - \tfrac{1}{2}n)\,!\; (n + m + 1)\,!}.$$

The coefficients $h_{m,n}$ are zero if $m < n$ and also if m and n differ by an odd integer. If $m = 0$ the only h which exists is

$$h_{0,0} = 2.$$

If $m = 1$ the only h which exists is

$$h_{1,1} = \tfrac{2}{3}.$$

Hence, when $m = 0$,

$$a_n e_n = 2\beta_{n,0},$$

and, when $m = 1$,

$$a_n e_n = \tfrac{2}{3}\beta_{n,1}.$$

Hence, for the expansion of unity, we have

$$1 = 2\left\{\frac{\mathrm{E}_0}{e_0} + \frac{\beta_{20}}{e_2}\mathrm{E}_2 + \frac{\beta_{40}}{e_4}\mathrm{E}_4 + \ldots\right\}. \qquad (18)$$

And, for the expansion of μ,

$$\mu = \tfrac{2}{3}\left\{\frac{\mathrm{E}_1}{e_1} + \frac{\beta_{31}}{e_3}\mathrm{E}_3 + \frac{\beta_{51}}{e_5}\mathrm{E}_5 + \ldots\right\}. \qquad (19)$$

Since the β's decrease rapidly even for moderately large values of κ, only a small number of terms in (18) or (19) is required, when κ is not too great. The e's also can be much simplified, even when κ is moderately large. The expansions (18) and (19) are very important.

In a similar manner it is found that

$$\mathrm{P}_m(\mu) = \frac{2}{2m+1}\left[\frac{\beta_{0,m}}{e_0}\mathrm{E}_0 + \frac{\beta_{2,m}}{e_2}\mathrm{E}_2 + \ldots\right].$$

Hence an arbitrary function expressed as a series of Legendre polynomials can be expressed as a series of the functions E_n.

For example,

$$\begin{aligned}\Phi &= \mathrm{A}_0\mathrm{P}_0 + \mathrm{A}_1\mathrm{P}_1 + \mathrm{A}_2\mathrm{P}_2 + \ldots \\ &= \frac{2\mathrm{E}_0}{e_0}\left\{\frac{\mathrm{A}_0}{1} + \frac{\mathrm{A}_2\beta_{02}}{5} + \frac{\mathrm{A}_4\beta_{04}}{9} + \ldots\right\} \\ &\quad + \ldots \\ &\quad + \frac{2\mathrm{E}_1}{e_1}\left\{\frac{\mathrm{A}_1}{3} + \frac{\mathrm{A}_3\beta_{13}}{7} + \frac{\mathrm{A}_5\beta_{15}}{11} + \ldots\right\} \\ &\quad + \ldots.\end{aligned}$$

Now a plane wave travelling parallel to the axis of z may be represented by $e^{i\rho\mu}$, and this can be expressed by Φ, following RAYLEIGH, if we write

$$A_n = (2n+1)\, i^n \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{1}{\rho^{\frac{1}{2}}} J_{n+\frac{1}{2}}(\rho).$$

Hence

$$\rho^{\frac{1}{2}} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \Phi = \frac{2E_0}{e_0} \{J_{\frac{1}{2}} - \beta_{02} J_{\frac{5}{2}} + \beta_{04} J_{\frac{9}{2}} - \ldots\} + \ldots .$$

When we come to develop the solution of (5) it will be found that this reduces to

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \Phi = 2 \sum_{n=0}^{n=\infty} i^n \frac{E_n H_n}{e_n},$$

where n has all positive integral values. This is the expansion of a plane wave in a series of the ellipsoidal functions.

The Functions E_n and H_n.

It is convenient in this place to deal with the development of the functions E_n and H_n concurrently. There are many ways in which the equations (4) and (5) can be transformed by substitution. Some of these will be considered, and the solutions obtained, the general method being the same in all cases. Rewrite equation (4) in the form

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dE_n}{d\mu}\right\} + n(n+1) E_n = \kappa^2 (1-\mu^2) E_n - a_n E_n . \quad . \quad . \quad (20)$$

Denote the left-hand side of (20) by ∇E_n, and the right-hand side by DE_n. The solution of (20) can be obtained in the following manner.

If P_s be a LEGENDRE polynomial of order s, there exists the recurrence formula

$$\mu P_s = \frac{s+1}{2s+1} P_{s+1} + \frac{s}{2s+1} P_{s-1},$$

from which $(1-\mu^2) P_s$ can be obtained at once. Hence, when P_s is substituted for E_n in DE_n we obtain

$$DP_s = L_s P_{s+2} + M_s P_s + N_s P_{s-2},$$

where

$$\left.\begin{aligned} L_s &= -\kappa^2 \frac{(s+1)(s+2)}{(2s+1)(2s+3)} \\ M_s &= \kappa^2 \frac{2(s^2+s-1)}{(2s+3)(2s-1)} - a_n \\ N_s &= -\kappa^2 \frac{s(s-1)}{(2s-1)(2s+1)} \end{aligned}\right\} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

If P_s be substituted for E_n in ∇E_n there results

$$\nabla P_s = \alpha_s P_s,$$

where

$$\alpha_s = (n - s)(n + s + 1). \qquad (22)$$

If, again, P_s be substituted for E_n in $(\nabla - D) E_n$ we have

$$(\nabla - D) P_s = - L_s P_{s+2} - A_s P_s - N_s P_{s-2}, \qquad (23)$$

where

$$A_s = M_s - \alpha_s.$$

Since the LEGENDRE function of the second kind, viz., Q_s, is subject to the same recurrence formulæ, P_s may be replaced by $AP_s + BQ_s$ where A and B are arbitrary constants.

The first transformation to be considered is that obtained by writing in (4)

$$E_n = e^{\kappa\mu} G_n.$$

The differential equation satisfied by G_n is easily found to be

$$\frac{d}{d\mu}\left\{\left(1 - \mu^2\right)\frac{dG_n}{d\mu}\right\} + n(n+1) G_n = -2\kappa\left\{\left(1 - \mu^2\right)\frac{dG_n}{d\mu} - \mu G_n\right\} - a_n G_n \qquad (24)$$

Denote the left-hand side of (24) by ∇G_n, and the right-hand side by DG_n.

Now we have the recurrence formulæ which give

$$(\mu^2 - 1)\frac{dP_s}{d\mu} + \mu P_s = (s+1)\mu P_s - sP_{s-1}$$

$$= \frac{(s+1)^2}{2s+1} P_{s+1} - \frac{s^2}{2s+1} P_{s-1}.$$

Hence in this case

$$DP_s = L_s P_{s+1} + M_s P_s + N_s P_{s-1},$$

where

$$\left.\begin{aligned} L_s &= 2\kappa\frac{(s+1)^2}{2s+1} \\ M_s &= -a_n \\ N_s &= -2\kappa\frac{s^2}{2s+1} \end{aligned}\right\}. \qquad (25)$$

Also

$$\nabla P_s = \alpha_s P_s$$

$$= (n - s)(n + s + 1) P_s. \qquad (26)$$

It must be pointed out here that the symbols used will not be altered for each transformation. This will not, however, cause any confusion. Although the functions H_n might be developed in a very similar manner in series of LEGENDRE functions of the second kind with imaginary argument, it is very much better to transform equation (5) with the object of obtaining solutions in series of Bessel functions. Writing, therefore,

$$H_n = \rho^{-\frac{1}{2}} R_n$$

equation (5) transforms into

$$\rho^2 \frac{d^2 R_n}{d\rho^2} + \rho \frac{dR_n}{d\rho} - (n+\tfrac{1}{2})^2 R_n + \rho^2 R_n$$
$$= -\frac{\kappa^2}{\rho^2}\left\{\rho^2 \frac{d^2 R_n}{d\rho^2} - \rho \frac{dR_n}{d\rho} + \tfrac{3}{4} R_n\right\} - \kappa^2 R_n + a_n R_n. \qquad (27)$$

Denote the left-hand side of (27) by ∇R_n, and the right-hand side by DR_n.

Now, if $J_m(\rho)$ be a Bessel function of order m,

$$\frac{d^2 J_m}{d\rho^2} = -\frac{1}{\rho}\frac{dJ_m}{d\rho} - \left(1 - \frac{m^2}{\rho^2}\right) J_m.$$

Also, from the recurrence formulæ,

$$2m\frac{J_m}{\rho^2} = \frac{1}{2(m-1)}\{J_{m-2} + J_m\} + \frac{1}{2(m+1)}\{J_m + J_{m+2}\},$$
$$\frac{1}{\rho}\frac{dJ_m}{d\rho} = \frac{1}{2(m-1)}\{J_{m-2} + J_m\} - \frac{m}{\rho^2} J_m.$$

If J_m be substituted for R_n in DR_n the latter can be expressed in terms of J_{m-2}, J_m, and J_{m+2}.

It is convenient, on account of the very frequent occurrence of the suffix $s+\frac{1}{2}$, to write

$$F_s = J_{s+\frac{1}{2}}.$$

Putting $m = s + \frac{1}{2}$ it is found easily that, if F_s be substituted for R_n in DR_n,

$$DF_s = L_s F_{s+2} + M_s F_s + N_s F_{s-2},$$

where

$$\left.\begin{aligned} L_s &= -\kappa^2 \frac{(s+1)(s+2)}{(2s+1)(2s+3)} \\ M_s &= -\kappa^2 \frac{2(s^2+s-1)}{(2s+3)(2s-1)} + a_n \\ N_s &= -\kappa^2 \frac{s(s-1)}{(2s-1)(2s+1)} \end{aligned}\right\} \qquad (28)$$

If F_s be substituted for R_n in ∇R_n there results

$$\nabla F_s = \alpha_s F_s,$$

where

$$\alpha_s = -(n-s)(n+s+1). \qquad (29)$$

If, again, F_s be substituted for R_n in $(\nabla - D) R_n$ we have

$$(\nabla - D) F_s = - L_s F_{s+2} - A_s F_s - N_s F_{s-2}, \qquad (30)$$

where

$$A_s = M_s - \alpha_s.$$

The similarity between (21)–(23) and (28)–(30) should be noted.

For another transformation of equation (5) we write

$$\rho^2 + \kappa^2 = \sigma^2.$$

When this substitution is effected the transformed equation is

$$\sigma^2 \frac{d^2 H_n}{d\sigma^2} + 2\sigma \frac{dH_n}{d\sigma} - n(n+1) H_n + \sigma^2 H_n = \kappa^2 \left\{ \frac{d^2 H_n}{d\sigma^2} + \frac{1}{\sigma}\frac{dH_n}{d\sigma} \right\} + a_n H_n. \qquad (31)$$

Putting

$$H_n = \sigma^{-\frac{1}{2}} S_n,$$

the equation satisfied by S_n is

$$\sigma^2 \frac{d^2 S_n}{d\sigma^2} + \sigma \frac{dS_n}{d\sigma} - (n + \tfrac{1}{2})^2 S_n + \sigma^2 S_n = \kappa^2 \left\{ \frac{d^2 S_n}{d\sigma^2} + \frac{1}{4\sigma^2} S_n \right\} + a_n S_n. \qquad (32)$$

As before, writing DS_n for the right-hand side and ∇S_n for the left-hand side of (32), and using the recurrence formulæ, we find that, the variable being σ,

$$DF_s = L_s F_{s+2} + M_s F_s + N_s F_{s-2},$$

where

$$\left.\begin{aligned} L_s &= \kappa^2 \frac{(s+1)^2}{(2s+1)(2s+3)} \\ M_s &= -\kappa^2 \frac{2(s^2+s-1)}{(2s+3)(2s-1)} + a_n \\ N_s &= \kappa^2 \frac{s^2}{(2s-1)(2s+1)} \end{aligned}\right\} \qquad (33)$$

Also

$$\nabla F_s = \alpha_s F_s = -(n-s)(n+s+1) F_s. \qquad (34)$$

It may be noted that, in the case of the function G_n, P_s may be replaced by

$$AP_s + BQ_s.$$

Also in the expansions of R_n and S_n, F_s may be replaced by

$$AJ_{s+\frac{1}{2}} + BJ_{-s-\frac{1}{2}},$$

where A and B are arbitrary constants.

The differential equations for E_n, G_n, R_n, and S_n can each be solved by one or other of two methods.

Solution by Continued Approximations.

Each of them may be written in the form

$$\nabla F - DF = 0. \qquad (35)$$

A first approximation to (35) is a solution of

$$\nabla F = 0.$$

Consider, as typical of all, $F = R_n$.

The first approximation is denoted by

$$F_n = U_0, \quad \text{say.}$$

If this first approximation be substituted in DR_n we obtain*

$$L_n F_{n+2} + M_n F_n + N_n F_{n-2} = DF_n = V_1, \text{ say.}$$

Write

$$W_1 = V_1 - \beta_1 F_n,$$

where

$$\beta_1 = M_n.$$

The reason for doing this is that the solution of the equation

$$\nabla R_n = F_n,$$

contains terms which cannot satisfy the conditions of the physical problem.

The process determines the value of a_n, which is the same for each of the functions E_n, G_n, R_n, and S_n.

A second approximation to (35) is

$$\frac{L_n}{\alpha_{n+2}} F_{n+2} + \frac{N_n}{\alpha_{n-2}} F_{n-2} = U_1, \text{ say.}$$

Substituting U_1 in DR_n we obtain

$$\frac{L_n}{\alpha_{n+2}} \{L_{n+2} F_{n+4} + M_{n+2} F_{n+2} + N_{n+2} F_n\}$$
$$+ \frac{N_n}{\alpha_{n-2}} \{L_{n-2} F_n + M_{n-2} F_{n-2} + N_{n-2} F_{n-4}\} = V_2, \text{ say.}$$

Write

$$W_2 = V_2 - \beta_2 F_n,$$

where

$$\beta_2 = \frac{L_n N_{n+2}}{\alpha_{n+2}} + \frac{L_{n-2} N_n}{\alpha_{n-2}}.$$

* In WHITTAKER and WATSON'S "Modern Analysis" this method is applied to the Differential Equation of MATHIEU'S functions.

A third approximation to (35) is

$$\frac{L_n L_{n+2}}{\alpha_{n+2}\,\alpha_{n+4}} F_{n+4} + \frac{L_n M_{n+2}}{\alpha^2{}_{n+2}} F_{n+2} + \frac{N_n M_{n-2}}{\alpha^2{}_{n-2}} F_{n-2} + \frac{N_n N_{n-2}}{\alpha_{n-2}\,\alpha_{n-4}} F_{n-4} = U_2, \text{ say.}$$

Proceeding in this way we obtain

$$\beta_3 = \frac{L_n M_{n+2} N_{n+2}}{\alpha^2{}_{n+2}} + \frac{N_n M_{n-2} L_{n-2}}{\alpha^2{}_{n-2}}.$$

Now, clearly,

$$R_n = \sum_{s=0} U_s,$$

and

$$\nabla \{\sum_{s=0} U_s\} = \sum_{s=1} W_s = \sum_{s=1} V_s - \sum_{s=1} \beta_s \,.\, F_n.$$

But $\sum_{s=1} V_s$ is the result of substituting

$$\sum_{s=0} U_s \quad \text{in} \quad DR_n.$$

Hence equation (35) is satisfied by

$$\sum_{s=0} U_s$$

if

$$\sum_{s=1} \beta_s = 0. \qquad (36)$$

Equation (36) determines the value of a_n. We shall assume for the expansion of R_n that

$$R_n = \ldots + \gamma_{n,\,n-2} F_{n-2} + F_n + \gamma_{n,\,n+2} F_{n+2} + \ldots . \qquad (37)$$

Then, as far as the sixth power of κ,

$$\left.\begin{aligned} \gamma_{n,\,n+2} &= \frac{L_n}{\alpha_{n+2}}\left\{1 + \frac{M_{n+2}}{\alpha_{n+2}} + \frac{M^2{}_{n+2}}{\alpha^2{}_{n+2}} + \frac{L_{n+2}}{\alpha_{n+2}} \cdot \frac{N_{n+4}}{\alpha_{n+4}}\right\} \\ \gamma_{n,\,n+4} &= \frac{L_n L_{n+2}}{\alpha_{n+2}\,\alpha_{n+4}}\left\{1 + \frac{M_{n+4}}{\alpha_{n+4}} + \frac{M_{n+2}}{\alpha_{n+2}}\right\} \\ \gamma_{n,\,n+6} &= \frac{L_n L_{n+2} L_{n+4}}{\alpha_{n+2}\,\alpha_{n+4}\,\alpha_{n+6}} \end{aligned}\right\} \qquad (38)$$

To obtain $\gamma_{n,\,n-2}$, etc., replace each L in (38) by N, and each N by L. Also replace each suffix $n + m$ by $n - m$. For example, to the sixth power of κ,

$$\gamma_{n,\,n-2} = \frac{N_n}{\alpha_{n-2}}\left\{1 + \frac{M_{n-2}}{\alpha_{n-2}} + \frac{M^2{}_{n-2}}{\alpha^2{}_{n-2}} + \frac{N_{n-2}}{\alpha_{n-2}} \cdot \frac{L_{n-4}}{\alpha_{n-4}}\right\}.$$

Finally for the evaluation of a_n to the sixth power of κ we have

$$M_n + \frac{L_n N_{n+2}}{\alpha_{n+2}}\left\{1 + \frac{M_{n+2}}{\alpha_{n+2}}\right\} + \frac{L_{n-2} N_n}{\alpha_{n-2}}\left\{1 + \frac{M_{n-2}}{\alpha_{n-2}}\right\} = 0. \qquad (39)$$

Now we have already assumed for the expansion of E_n that

$$E_n = \ldots + \beta_{n,\,n-2} P_{n-2} + P_n + \beta_{n,\,n+2} P_{n+2} + \ldots .$$

If we compare the values of the L's, M's, and N's, and also of the α's, in the development of E_n with those obtained in the development of R_n, it is at once apparent that the expressions within brackets in (38) are identical. The equation for the determination of a_n is, as it should be, the same in both cases.

We have then

$$\beta_{n,\,n+2} = -\gamma_{n,\,n+2}$$

$$\beta_{n,\,n+4} = +\gamma_{n,\,n+4}, \text{ etc.}$$

Hence

$$E_n = \ldots - \gamma_{n,\,n-2} P_{n-2} + P_n - \gamma_{n,\,n+2} P_{n+2} + \ldots .$$

For both E_n and R_n, therefore, attention need only be paid to the development of the γ's.

With regard to the function G_n, although it represents an elegant solution for small values of κ, its coefficients converge too slowly for large values of κ to be of much practical use. It will be noticed in (21) that for the expansion of E_n in Legendre functions and in (28) that for the expansion of R_n in Bessel functions the indicial equation $N_s = 0$ has the roots $s = 0$ and $s = 1$.

For the expansion of S_n, however, in Bessel functions it will be seen in (33) that the indicial equation $N_s = 0$ has only one root.

The functions S_n will not, therefore, be retained, although they are convergent for small values of σ.

It has been seen, further, that, in the expansion of R_n, F_s may be replaced by

$$AJ_{s+\frac{1}{2}} + BJ_{-s-\frac{1}{2}}.$$

But the latter function $J_{-s-\frac{1}{2}}$ is divergent for small values of ρ. Hence we have still to find a solution competent to represent the divergent wave. Before considering this question, however, it is advisable to devote some attention to the coefficients (38).

The Determination of a_n. First Approximations.

We have from (28), in the first place,

$$M_s = -\tfrac{1}{2}\kappa^2 \frac{4s^2 + 4s - 4}{4s^2 + 4s - 3} + a_n. \qquad (40)$$

Equation (40) may be written

$$M_s = -\tfrac{1}{2}\kappa^2 \left\{1 - \frac{1}{(2s+3)(2s-1)}\right\} + a_n$$

$$= m_s + a_n, \quad \text{say.}$$

Hence, from (39)

$$m_n + a_n + b_n = 0,$$

where

$$b_n = \frac{L_n N_{n+2}}{\alpha_{n+2}}\left\{1 + \frac{M_{n+2}}{\alpha_{n+2}}\right\} + \frac{L_{n-2}N_n}{\alpha_{n-2}}\left\{1 + \frac{M_{n-2}}{\alpha_{n-2}}\right\}.$$

It follows that

$$a_n\left\{1 + \frac{L_n N_{n+2}}{\alpha^2_{n+2}} + \frac{L_{n-2}N_n}{\alpha^2_{n-2}}\right\} + m_n + \frac{L_n N_{n+2}}{\alpha_{n+2}}\left\{1 + \frac{m_{n+2}}{\alpha_{n+2}}\right\} + \frac{L_{n-2}N_n}{\alpha_{n-2}}\left\{1 + \frac{m_{n-2}}{\alpha_{n-2}}\right\} = 0.$$

Hence, as far as κ^6,

$$a_n = -m_n + \frac{L_n N_{n+2}}{\alpha_{n+2}}\left\{\frac{m_n - m_{n+2}}{\alpha_{n+2}} - 1\right\} + \frac{L_{n-2}N_n}{\alpha_{n-2}}\left\{\frac{m_n - m_{n-2}}{\alpha_{n-2}} - 1\right\}.$$

Now

$$\frac{m_n - m_s}{\alpha_s} = \frac{2\kappa^2}{(2s+3)(2s-1)(2n+3)(2n-1)}.$$

Hence a_n can be obtained as far as κ^6.

The calculation of a_n for large values of κ is exceedingly laborious, but since it is vital to the success of the theory we shall return to it when considering the method of series.

The value of b_n is, accordingly,

$$b_n = \kappa^4 \frac{(n+1)^2(n+2)^2}{2(2n+1)(2n+3)^3(2n+5)}\left\{1 - \frac{2\kappa^2}{(2n+7)(2n+3)^2(2n-1)}\right\}$$
$$-\kappa^4 \frac{n^2(n-1)^2}{2(2n+1)(2n-1)^3(2n-3)}\left\{1 - \frac{2\kappa^2}{(2n+3)(2n-1)^2(2n-5)}\right\}. \quad (41)$$

Further, it follows easily that

$$\frac{M_s}{\alpha_s} = \frac{m_s + a_n}{\alpha_s} = -\frac{m_n - m_s}{\alpha_s} - \frac{b_n}{\alpha_s}$$
$$= -\frac{2\kappa^2}{(2s+3)(2s-1)(2n+3)(2n-1)} + \frac{b_n}{(n-s)(n+s+1)}, \quad (42)$$

$$\frac{L_s}{\alpha_{s+2}} = \kappa^2 \frac{(s+1)(s+2)}{(2s+1)(2s+3)(n-s-2)(n+s+3)}, \quad (43)$$

$$\frac{N_s}{\alpha_{s-2}} = \kappa^2 \frac{s(s-1)}{(2s+1)(2s-1)(n-s+2)(n+s-1)}, \quad (44)$$

$$\frac{L_s N_{s+2}}{\alpha_s \alpha_{s+2}} = \kappa^4 \frac{(s+1)^2(s+2)^2}{(2s+1)(2s+3)^2(2s+5)(n-s)(n+s+1)(n-s-2)(n+s+3)}. \quad (45)$$

From the expressions (41)–(45) the values of the coefficients γ can be written down at once, and will be used later.

The Solution by the Method of Series.

By this method it is possible to examine the value of a_n for moderately large values of κ, and it is of particular value in the solution for the divergent wave. Assume for the expression of R_n, as before,

$$R_n = \ldots + \gamma_{n,\,n-2} F_{n-2} + F_n + \gamma_{n,\,n+2} F_{n+2} + \ldots.$$

Substituting in the differential equation (27) we must have

$$\begin{aligned} \ldots &+ \gamma_{n,\,n-2}\{L_{n-2}F_n + A_{n-2}F_{n-2} + N_{n-2}F_{n-4}\} \\ &+ \{L_nF_{n+2} + A_nF_n + N_nF_{n-2}\} \\ &+ \gamma_{n,\,n+2}\{L_{n+2}F_{n+4} + A_{n+2}F_{n+2} + N_{n+2}F_n\} + \ldots = 0. \quad (46) \end{aligned}$$

The equation to determine a_n is

$$\gamma_{n,\,n-2}L_{n-2} + A_n + \gamma_{n,\,n+2}N_{n+2} = 0. \quad (47)$$

The equations obtained by equating to zero the coefficients of F_{n-4}, F_{n-2}, F_{n+2}, F_{n+4}, etc., in (46) must also hold.

It is convenient to denote L_n/α_{n+2} by L'_n, A_{n+2}/α_{n+2} by A'_{n+2}, N_{n+4}/α_{n+2} by N'_{n+4}, L_{n+2}/α_{n+4} by L'_{n+2}, etc.

Denote by Δ_m the infinite determinant

$$\begin{vmatrix} A'_{n+m+2} & N'_{n+m+4} & 0 & \\ L'_{n+m+2} & A'_{n+m+4} & N'_{n+m+6} & \\ 0 & L'_{n+m+4} & A'_{n+m+6} & \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}.$$

Then

$$\gamma_{n,\,n+m} \cdot \Delta_0 = (-)^{\frac{1}{2}m} L'_n L'_{n+2} \ldots L'_{n+m-2} \cdot \Delta_m. \quad (48)$$

It is also found that if we denote by Δ'_m the determinant

$$\begin{vmatrix} A'_{n-m-2} & L'_{n-m-4} & 0 & \\ N'_{n-m-2} & A'_{n-m-4} & L'_{n-m-6} & \\ 0 & N'_{n-m-4} & A'_{n-m-6} & \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix},$$

then

$$\gamma_{n,\,n-m} \cdot \Delta'_0 = (-)^{\frac{1}{2}m} N'_n N'_{n-2} \ldots N'_{n-m+2} \cdot \Delta'_m. \quad (49)$$

The determinant Δ'_m is finite.

The value of $\Delta_0 \longrightarrow 1$ as n increases.

So also does the value of Δ'_0.

The value of $\Delta_m \longrightarrow 1$ as m increases, for all values of n.

We commence by expanding Δ_0.

We have

$$\Delta_0 = A'_{n+2}\Delta_2 - L'_{n+2}N'_{n+4}\Delta_4,$$
$$\Delta_2 = A'_{n+4}\,\Delta_4 - L'_{n+4}N'_{n+6}\Delta_6,$$

and so on.

If it be desired to expand Δ_0/Δ_2 as far as the sixth power of κ, the last term in the foregoing expression for Δ_2 is not required. Moreover, $L'_{n+4}N'_{n+6}$ is exceedingly small for all values of n if κ be not greater than 5.

Hence, approximately,

$$\Delta_2 = A'_{n+4}\Delta_4.$$

In like manner,

$$\Delta'_0 = A'_{n-2}\Delta'_2 - L'_{n-4}N'_{n-2}\Delta'_4,$$
$$\Delta'_2 = A'_{n-4}\Delta'_4.$$

The coefficient N'_{n-2} vanishes when $n < 4$.

The above equations are sufficient to enable us to expand b_n in powers of κ^2 up to κ^{10}.

Further Consideration of a_n.

We have found that

$$\gamma_{n,\,n+2} = -\frac{\Delta_2}{\Delta_0}L'_n,$$

$$\gamma_{n,\,n-2} = -\frac{\Delta'_2}{\Delta'_0}N'_n,$$

and

$$A_n = M_n = -b_n.$$

Hence the equation to determine b_n is, from (47),

$$\frac{N'_nL_{n-2}}{\Delta_0'/\Delta'_2} + b_n + \frac{L'_nN_{n+2}}{\Delta_0/\Delta_2} = 0, \quad \ldots \ldots \ldots \ldots \quad (50)$$

where

$$N'_nL_{n-2} = -\kappa^4\frac{n^2(n-1)^2}{2(2n-1)^3(2n-3)(2n+1)}$$

and

$$L'_nN_{n+2} = +\kappa^4\frac{(n+1)^2(n+2)^2}{2(2n+3)^3(2n+5)(2n+1)}.$$

Further,

$$\left.\begin{aligned}\frac{\Delta_0}{\Delta_2} &= A'_{n+2} - \frac{L'_{n+2}N'_{n+4}}{A'_{n+4}},\\ \frac{\Delta'_0}{\Delta'_2} &= A'_{n-2} - \frac{L'_{n-4}N'_{n-2}}{A'_{n-4}}.\end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \quad (51)$$

It is convenient to write

$$A'_s = -M'_s + \frac{b_n}{(n-s)(n+s+1)} - 1,$$

where

$$M'_s = \frac{2\kappa^2}{(2s+3)(2s-1)(2n+3)(2n-1)}.$$

The equations (51) are sufficient for the expansion of Δ_0/Δ_2 and Δ'_0/Δ'_2 up to and including the sixth power of κ.

Stopping at this power, then, since the lowest power of κ in b_n is the fourth, we may clearly omit the terms containing b_n in A'_{n+4} and A'_{n-4}.

Inserting the value of b_n, which is given in (41) to the sixth power of κ, in A'_{n+2} and A'_{n-2}, and substituting these latter in (51), the expansion of b_n to the tenth power of κ is obtained by means of (50).

While the expansions thus found are valid for all values of n, so long as κ is not much greater than 3, it is soon discovered that, as κ approaches the value 5 when n is small, the series for b_n in ascending powers of κ^2 appears to become useless as an approximation. This makes it imperative to reconsider the method by which the value of a_n has been derived.

The equations (50) and (51) are sufficient to evaluate b_n with tolerable accuracy for values of κ as great as five and for all values of n, but not by means of an expansion in ascending powers of κ^2.

The factor N'_{n-2} in the last term of (51) is zero when $n < 4$, and for all values of n the last term on the right-hand side of the equation for Δ'_0/Δ'_2 may be neglected. This may be verified for any particular case. The last term on the right-hand side of the equation for Δ_0/Δ_2 is small, but it cannot be neglected for all values of n. Since, however, the denominator of that term does not differ greatly from unity, a simplification can be made. It is assumed that the denominator, viz., $-A'_{n+4}$, differs little from unity for all values of n and that, without serious error, we may put $-A'_{n+6}$, $-A'_{n+8}$, etc., each equal to unity. These assumptions can be justified subsequently.

It follows, then, that, since $M'_{n+4} + b_n/4(2n+5)$ is assumed to be small,

$$\frac{\Delta_0}{\Delta_2} = -1 - M'_{n+2} - \frac{b_n}{2(2n+3)} + L'_{n+2}N'_{n+4}\left\{1 - M'_{n+4} - \frac{b_n}{4(2n+5)}\right\}$$

and

$$\frac{\Delta'_0}{\Delta'_2} = -1 - M'_{n-2} + \frac{b_n}{2(2n-1)}.$$

For brevity we shall write

$$\alpha = 1 + M'_{n+2} - L'_{n+2}N'_{n+4}(1 - M'_{n+4}),$$

$$\beta = \frac{1}{2(2n+3)} + \frac{L'_{n+2}N'_{n+4}}{4(2n+5)},$$

$$\alpha' = 1 + M'_{n-2}, \qquad \beta' = \frac{1}{2(2n-1)},$$

$$A = L'_nN_{n+2}, \qquad A' = N'_nL_{n-2}.$$

Then (50) may be written in the form

$$\frac{A'}{-\alpha' + \beta' b_n} + b_n - \frac{A}{\alpha + \beta b_n} = 0. \quad \ldots \ldots \ldots \quad (52)$$

Now, although A′ is zero when $n < 2$, it is the most important term in the evaluation of b_n when > 1.

Taking this case first, when $n > 1$, it may be assumed by inspection that $\alpha + \beta b_n$ changes slowly, since it does not differ greatly from unity. Let us therefore put

$$b_n = x + \rho$$

where x satisfies the equation

$$A' + (x - A)(\beta' x - \alpha') = 0. \quad \ldots \ldots \ldots \quad (53)$$

The equation satisfied by ρ is, then,

$$\frac{A'}{\lambda' + \beta'\rho} + x + \rho - \frac{A}{\lambda + \beta\rho} = 0,$$

where

$$\lambda = \alpha + \beta x, \qquad \lambda' = -\alpha' + \beta' x.$$

It may now be assumed that $\beta\rho/\lambda$ is a small quantity whose square may be neglected in comparison with unity. With this assumption, which can be justified subsequently, we may write

$$\frac{1}{\lambda + \beta\rho} = \frac{1}{\lambda}\left(1 - \frac{\beta\rho}{\lambda}\right).$$

The equation to determine ρ is then a quadratic, which may be solved in terms of x.

Consider now equation (53) which gives the most important part of b_n.

The solution of this equation depends upon the value of the radical

$$\left\{1 - 4\beta' \cdot \frac{A\alpha' + A'}{(A\beta' + \alpha')^2}\right\}^{\frac{1}{2}}.$$

If this radical be written

$$(1 - \gamma)^{\frac{1}{2}},$$

then the expansion of the radical depends upon the magnitude of γ, and the assumption made when developing b_n in ascending powers of κ^2 is that this expansion is justified. The convergence of the series for b_n may become very slow for moderately large values of κ.

This is very clearly exemplified when $n = 2$.

When $n < 2$ the equation to determine b_n is

$$b_n(\beta b_n + \alpha) - A = 0,$$

the solution being

$$b_n = -\frac{\alpha}{2\beta} + \left(\frac{\alpha^2}{4\beta^2} + \frac{A}{\beta}\right)^{\frac{1}{2}}, \quad \ldots \ldots \ldots \quad (54)$$

the positive sign being used with the radical since b_n must vanish with κ. It is now clear why the expansion of b_n in a series of ascending powers of κ^2 fails. For, consider the equation for α, viz.,

$$\alpha = 1 + M'_{n+2} - L'_{n+2} N'_{n+4} (1 - M'_{n+4}).$$

When $n = 0$

$$\alpha = 1 - \frac{2\kappa^\epsilon}{7 . 9} - \frac{9 . 16\kappa^4}{5 . 7 . 7 . 9 . 2 . 3 . 4 . 5}\left(1 + \frac{2\kappa^2}{11 . 7 . 3}\right),$$

and this vanishes when κ has a value a little less than five.

For such a value of κ the expansion of the radical in (54) is impossible. Having obtained expressions for b_n when κ is as great as five, we may now find the coefficients γ.

With the degree of accuracy required,

$$\frac{\Delta_0}{\Delta_4} = \frac{\Delta_0}{\Delta_2} . \frac{\Delta_2}{\Delta_4} = \frac{\Delta_0}{\Delta_2} . A'_{n+4},$$

$$\frac{\Delta_0}{\Delta_6} = -\frac{\Delta_0}{\Delta_2} . A'_{n+4}, \quad \text{etc.},$$

since we have assumed that

$$A'_{n+6} = A'_{n+8} = \ldots = -1.$$

Similar expressions can be written down for Δ'_0/Δ'_m.

The γ's are then obtained at once from (48) and (49). In fact,

$$-\beta_{n,\,n+2} = \gamma_{n,\,n+2} = -L'_n \frac{\Delta_2}{\Delta_0},$$

$$\beta_{n,\,n+4} = \gamma_{n,\,n+4} = \frac{L'_n L'_{n+2}}{A'_{n+4}} . \frac{\Delta_2}{\Delta_0} \ldots, \text{ etc.}$$

where

$$1/A'_{n+4} = -1 + M'_{n+4} + b_n/4\,(2n + 5).$$

Similar expressions can be written down for $\gamma_{n,\,n-2}$, etc.

The Expansion of E_0 and E_1.

When $n = 0$ or 1

$$\frac{\Delta_2}{\Delta_0} = -\frac{b_n}{L'_n N_{n+2}}.$$

Hence, when $n = 0$ or 1,

$$E_n = P_n - \frac{b_n}{N_{n+2}} P_{n+2} - \frac{b_n L'_{n+2}}{N_{n+2} A'_{n+4}} (P_{n+4} - L'_{n+4} P_{n+6} + \ldots).$$

Since $L'_{n+4} L'_{n+6}$ may be neglected if κ be not greater than five or thereabout, the terms P_{n+8} and higher may be omitted.

We shall consider the case $n = 0$ in detail. In this case

$$N_{n+2} = -\kappa^2 \cdot \frac{2 \cdot 1}{3 \cdot 5}$$

$$L'_{n+2} = -\kappa^2 \frac{3 \cdot 4}{5 \cdot 7 \cdot 4 \cdot 5}$$

$$L'_{n+4} = -\kappa^2 \frac{5 \cdot 6}{9 \cdot 11 \cdot 6 \cdot 7}$$

$$-\frac{1}{A'_{n+4}} = 1 + \frac{2\kappa^2}{11 \cdot 7 \cdot 3 \cdot 1} - \frac{b_0}{20}.$$

Hence

$$E_0 = P_0 + \frac{3 \cdot 5}{2\kappa^2} b_0 P_2 + b_0 \left(1 + \frac{2\kappa^2}{11 \cdot 7 \cdot 3} - \frac{b_0}{20}\right) \frac{3 \cdot 3 \cdot 5}{5 \cdot 7 \cdot 5 \cdot 2} \left(P_4 + \frac{5\kappa^2}{11 \cdot 9 \cdot 7} P_6\right). \quad (55)$$

The value of b_0 is given by (54), where, as already stated,

$$\alpha = 1 - \frac{2\kappa^2}{7 \cdot 9} - \frac{9 \cdot 16\kappa^4}{5 \cdot 7 \cdot 7 \cdot 9 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \left(1 + \frac{2\kappa^2}{11 \cdot 7 \cdot 3}\right).$$

Further,

$$\beta = \frac{1}{6} + \frac{1}{20} \cdot \frac{9 \cdot 16\kappa^4}{5 \cdot 7 \cdot 7 \cdot 9 \cdot 2 \cdot 3 \cdot 4 \cdot 5}, \qquad A = \frac{4\kappa^4}{2 \cdot 27 \cdot 5}.$$

In the expression (55), for values of κ up to 5, it may be shown that, without serious error, we may omit

$$\frac{2\kappa^2}{11 \cdot 7 \cdot 3} - \frac{b_0}{20},$$

and put $\beta = 1/6$.

Some Remarks upon the Approximations.

A few remarks should be made here with regard to the nature of the approximations. In the determination of b_n, whose expression in terms of κ and n appears to be one of very great complexity, attention is in the first place directed to the terms which constitute the diagonals adjacent to the central diagonal of the determinants involved.

These terms are independent of b_n, they always occur in products of pairs, and these products diminish very rapidly. A general discussion would be very laborious, and it is almost self-evident in the nature of the problem that the exact mathematical representation of the mutual interference of the scattered waves for large values of κ must be very complicated.

It has been found that the evaluation of the integral equation and the method of successive approximations both lead to expressions in series of ascending powers of κ^2 for the coefficients. These series answer satisfactorily for all values of n so long as κ is not much greater than 3, but when it was attempted to use the series for larger values of κ they appeared to become divergent, at all events for small values of n.

The reason for this became clear when the differential equation was solved by the method of series. By this method it was found possible to obtain the value of b_n with ample accuracy so long as κ was not much greater than 5. Having found b_n the determination of the coefficients is straightforward, but it must be observed that each step in the approximations should be properly justified and this is only possible after b_n has been found.

By this procedure the nature of the approximations in the foregoing theory has been examined.

It is of interest to note that, when $\kappa = \pi = 3 \cdot 1416$, the corresponding wave-length is equal to the diameter of the focal circle of the ellipsoidal co-ordinates. Hence, when dealing with the problem of the circular disc or the circular aperture in a plane screen, when the diameter is equal to a wave-length or less than a wave-length, the coefficients in the expansion of H_n may be obtained most readily by an expansion in ascending powers of κ^2.

The Divergent Wave.

A solution of the wave equation suitable to represent a divergent wave has not yet been obtained. This problem is troublesome but, on account of its importance, we must now proceed to its consideration.

For this purpose we require a second solution of equation (5), for it will be observed that the Bessel functions $J_{-s-\frac{1}{2}}(\rho)$ are inadmissible for small values of ρ.

In equation (5) write

$$H_n = uv + w.$$

Upon substitution we have

$$u\left[\frac{d}{d\rho}\left\{(\rho^2+\kappa^2)\frac{dv}{d\rho}\right\} - \{n(n+1) + a_n - (\rho^2+\kappa^2)\}v\right]$$
$$+ \frac{d}{d\rho}\left\{(\rho^2+\kappa^2)\frac{dw}{d\rho}\right\} - \{n(n+1) + a_n - (\rho^2+\kappa^2)\}w$$
$$+ (\rho^2+\kappa^2)\left(2\frac{du}{d\rho}\cdot\frac{dv}{d\rho} + v\frac{d^2u}{d\rho^2}\right) + 2\rho v\frac{du}{d\rho} = 0. \qquad (56)$$

The coefficient of v in the last line of (56) is

$$(\rho^2 + \kappa^2)\frac{d^2u}{d\rho^2} + 2\rho\frac{du}{d\rho}. \qquad (57)$$

Let u be determined so that this is zero.

If (57) be equated to zero the required solution is found to be

$$u = \cot^{-1} \rho/\kappa. \qquad (58)$$

The value of $du/d\rho$ is, accordingly, given by

$$(\rho^2 + \kappa^2)\, du/d\rho = -\kappa. \qquad (59)$$

If, then, v is a solution of the first line of (56) when equated to zero, it follows that

$$\frac{d}{d\rho}\left\{(\rho^2 + \kappa^2)\frac{dw}{d\rho}\right\} - \{n(n+1) + a_n - (\rho^2 + \kappa^2)\}\, w = 2\kappa\frac{dv}{d\rho}. \qquad (60)$$

Now the value of v is, in our previous notation,

$$\rho^{-\frac{1}{2}} R_n,$$

and we shall write

$$w = \rho^{-\frac{1}{2}} T_n.$$

Substituting in (60) that equation transforms into

$$\rho^2\frac{d^2T_n}{d\rho^2} + \rho\frac{dT_n}{d\rho} - (n+\tfrac{1}{2})^2 T_n + \rho^2 T_n$$
$$= -\frac{\kappa^2}{\rho^2}\left\{\rho^2\frac{d^2T_n}{d\rho^2} - \rho\frac{dT_n}{d\rho} + \tfrac{3}{4}T_n\right\} - \kappa^2 T_n + a_n T_n + 2\kappa\left(\frac{dR_n}{d\rho} - \frac{R_n}{2\rho}\right). \qquad (61)$$

If now $J_{s+\frac{1}{2}}$ ($= F_s$) be substituted for R_n in the last term on the right-hand side of (61) that term becomes

$$2\kappa\left(\frac{dJ_{s+\frac{1}{2}}}{d\rho} - \frac{J_{s+\frac{1}{2}}}{2\rho}\right). \qquad (62)$$

Now, the recurrence formulæ give

$$\frac{2(s+\frac{1}{2})\,J_{s+\frac{1}{2}}}{\rho} = J_{s-\frac{1}{2}} + J_{s+\frac{3}{2}}$$

and

$$\frac{dJ_{s+\frac{1}{2}}}{d\rho} = J_{s-\frac{1}{2}} - \frac{(s+\frac{1}{2})\,J_{s+\frac{1}{2}}}{\rho}.$$

Hence (62) reduces to

$$\frac{2\kappa}{2s+1}\{sJ_{s-\frac{1}{2}} - (s+1)\,J_{s+\frac{3}{2}}\}$$

or

$$\frac{2\kappa}{2s+1}\{sF_{s-1} - (s+1)\,F_{s+1}\}. \qquad (63)$$

Hence, when R_n is replaced by its expansion in the last term on the right-hand side of (61) that term becomes

$$\begin{aligned}
& \cdots\cdots\cdots\cdots\cdots\cdots \\
&+ \gamma_{n, n-4} \cdot \frac{2\kappa}{2n-7}\{(n-4)\,F_{n-5} - (n-3)\,F_{n-3}\} \\
&+ \gamma_{n, n-2} \;\frac{2\kappa}{2n-3}\{(n-2)\,F_{n-3} - (n-1)\,F_{n-1}\} \\
&+ \qquad \frac{2\kappa}{2n+1}\{nF_{n-1} - (n+1)\,F_{n+1}\} \\
&+ \gamma_{n, n+2} \;\frac{2\kappa}{2n+5}\{(n+2)\,F_{n+1} - (n+3)F_{n+3}\} \\
&+ \gamma_{n, n+4} \;\frac{2\kappa}{2n+9}\{(n+4)\,F_{n+3} - (n+5)\,F_{n+5}\} \\
& \cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

It is convenient to write

$$\left.\begin{aligned}
B_{n, n-5} &= 2\kappa\,\gamma_{n, n-4}\frac{n-4}{2n-7} \\
B_{n, n-3} &= 2\kappa\left\{\gamma_{n, n-2}\frac{n-2}{2n-3} - \gamma_{n, n-4}\frac{n-3}{2n-7}\right\} \\
B_{n, n-1} &= 2\kappa\left\{\frac{n}{2n+1} + \gamma_{n, n-2}\frac{n-1}{2n-3}\right\} \\
B_{n, n+1} &= 2\kappa\left\{\gamma_{n, n+2}\frac{n+2}{2n+5} - \frac{n+1}{2n+1}\right\} \\
B_{n, n+3} &= 2\kappa\left\{\gamma_{n, n+4}\frac{n+4}{2n+9} - \gamma_{n, n+2}\frac{n+3}{2n+5}\right\} \\
B_{n, n+5} &= -\,2\kappa\,\gamma_{n, n+4}\frac{n+5}{2n+9}.
\end{aligned}\right\} \quad \ldots\ldots \quad (64)$$

Then, when R_n is replaced by its expansion in the last term on the right-hand side of (61) that term becomes

$$\ldots + B_{n, n-5}F_{n-5} + B_{n, n-3}F_{n-3} + B_{n, n-1}F_{n-1} + B_{n, n+1}F_{n+1} + B_{n, n+3}F_{n+3} + \ldots, \quad (65)$$

in which the coefficients B are known. Let us now temporarily replace the term

$$2\kappa\left\{\frac{dR_n}{d\rho} - \frac{R_n}{2\rho}\right\} \text{ in (61) by } F_s.$$

The solution of the resulting equation can then be obtained in the following manner by successive approximation, the terms of (61) containing κ^2 being omitted when obtaining the first approximation. The first approximation is

$$F_s/\alpha_s,$$

where, as before,

$$\alpha_s = -(n-s)(n+s+1).$$

The second approximation is

$$\frac{1}{\alpha_s}\left\{\frac{L_s}{\alpha_{s+2}}F_{s+2} + \frac{M_s}{\alpha_s}F_s + \frac{N_s}{\alpha_{s-2}}F_{s-2}\right\},$$

where L_s, M_s, and N_s are given by (28).

The third approximation is

$$\frac{L_s}{\alpha_s\alpha_{s+2}}\left\{\frac{L_{s+2}}{\alpha_{s+4}}F_{s+4} + \frac{M_{s+2}}{\alpha_{s+2}}F_{s+2} + \frac{N_{s+2}}{\alpha_s}F_s\right\}$$
$$+\frac{M_s}{\alpha^2_s}\left\{\frac{L_s}{\alpha_{s+2}}F_{s+2} + \frac{M_s}{\alpha_s}F_s + \frac{N_s}{\alpha_{s-2}}F_{s-2}\right\}$$
$$+\frac{N_s}{\alpha_s\alpha_{s-2}}\left\{\frac{L_{s-2}}{\alpha_s}F_s + \frac{M_{s-2}}{\alpha_{s-2}}F_{s-2} + \frac{N_{s-2}}{\alpha_{s-4}}F_{s-4}\right\},$$

and so on.

Corresponding, therefore, to F_s the solution of (61) may be denoted by I_s, where

$$\alpha_s I_s = \ldots + c_{s,\,s-4}F_{s-4} + c_{s,\,s-2}F_{s-2} + c_{s,\,s}F_s + c_{s,\,s+2}F_{s+2} + c_{s,\,s+4}F_{s+4} + \ldots, \quad (66)$$

and

$$\left.\begin{aligned}
c_{s,\,s} &= 1 + \frac{M_s}{\alpha_s} + \frac{L_sN_{s+2}}{\alpha_s\alpha_{s+2}} + \frac{M_s^2}{\alpha_s^2} + \frac{L_{s-2}N_s}{\alpha_{s-2}\alpha_s} + \ldots \\
c_{s,\,s+2} &= \frac{L_s}{\alpha_{s+2}}\left\{1 + \frac{M_{s+2}}{\alpha_{s+2}} + \frac{M_s}{\alpha_s}\right\} + \ldots \\
c_{s,\,s+4} &= \frac{L_sL_{s+2}}{\alpha_{s+2}\alpha_{s+4}} + \ldots \\
c_{s,\,s-2} &= \frac{N_s}{\alpha_{s-2}}\left\{1 + \frac{M_s}{\alpha_s} + \frac{M_{s-2}}{\alpha_{s-2}}\right\} + \ldots \\
c_{s,\,s-4} &= \frac{N_sN_{s-2}}{\alpha_{s-2}\alpha_{s-4}} + \ldots.
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (67)$$

The solution of (61) is, then, clearly

$$T_n = \ldots + B_{n,\,n-5}I_{n-5} + B_{n,\,n-3}I_{n-3} + B_{n,\,n-1}I_{n-1}$$
$$+ B_{n,\,n+1}I_{n+1} + B_{n,\,n+3}I_{n+3} + B_{n,\,n+5}I_{n+5} + \ldots. \quad . \quad . \quad (68)$$

This in turn may be expressed in the form

$$T_n = \ldots + \varepsilon_{n,\,n-5}F_{n-5} + \varepsilon_{n,\,n-3}F_{n-3} + \varepsilon_{n,\,n-1}F_{n-1}$$
$$+ \varepsilon_{n,\,n+1}F_{n+1} + \varepsilon_{n,\,n+3}F_{n+3} + \varepsilon_{n,\,n+5}F_{n+5} + \ldots. \quad . \quad . \quad . \quad (69)$$

We shall now proceed to obtain the coefficients of κ^2 in I_{n+1} and I_{n-1}, and of κ^3 in the coefficients B, so that we may obtain the values of the coefficients ε as far as the third power of κ.

By means of (28) and (67) we have

$$\left.\begin{aligned} c_{n+1,\,n+1} &= 1 - \frac{2\kappa^2}{(2n+5)(2n+1)(2n+3)(2n-1)} \\ c_{n+1,\,n+3} &= -\kappa^2 \frac{(n+2)(n+3)}{(2n+3)(2n+5)\,3\,(2n+4)} \\ c_{n+1,\,n-1} &= \kappa^2 \frac{n(n+1)}{(2n+3)(2n+1)\,1\,(2n)} \\ c_{n-1,\,n-1} &= 1 - \frac{2\kappa^2}{(2n+1)(2n-3)(2n+3)(2n-1)} \\ c_{n-1,\,n+1} &= -\kappa^2 \frac{n(n+1)}{(2n-1)(2n+1)\,1\,(2n+2)} \\ c_{n-1,\,n-3} &= \kappa^2 \frac{(n-1)(n-2)}{(2n-1)(2n-3)\,3\,(2n-2)}. \end{aligned}\right\} \quad \ldots \quad (70)$$

We require also

$$\begin{aligned} \gamma_{n,\,n-2} &= \frac{N_n}{\alpha_{n-2}} = \kappa^2 \frac{n(n-1)}{(2n+1)(2n-1)^2 2} \\ \gamma_{n,\,n+2} &= \frac{L_n}{\alpha_{n+2}} = -\kappa^2 \frac{(n+1)(n+2)}{(2n+1)(2n+3)^2 2}. \end{aligned} \quad \ldots\ldots\ldots \quad (71)$$

Now, from (68), the solution of (61) up to the third power of κ is clearly

$$\begin{aligned} T_n = B_{n,\,n-3}\frac{F_{n-3}}{\alpha_{n-3}} &+ \frac{B_{n,\,n-1}}{\alpha_{n-1}}\{c_{n-1,\,n-3}F_{n-3} + c_{n-1,\,n-1}F_{n-1} + c_{n-1,\,n+1}F_{n+1}\} \\ &+ \frac{B_{n,\,n+1}}{\alpha_{n+1}}\{c_{n+1,\,n-1}F_{n-1} + c_{n+1,\,n+1}F_{n+1} + c_{n+1,\,n+3}F_{n+3}\} \\ &+ B_{n,\,n+3}\frac{F_{n+3}}{\alpha_{n+3}}. \end{aligned} \quad \ldots\ldots\ldots\ldots\ldots \quad (72)$$

Hence

$$\begin{aligned} \varepsilon_{n,\,n-3} &= \frac{B_{n,\,n-3}}{\alpha_{n-3}} + \frac{B_{n,\,n-1}}{\alpha_{n-1}} c_{n-1,\,n-3} \\ \varepsilon_{n,\,n-1} &= \frac{B_{n,\,n-1}}{\alpha_{n-1}} c_{n-1,\,n-1} + \frac{B_{n,\,n+1}}{\alpha_{n+1}} c_{n+1,\,n-1} \\ \varepsilon_{n,\,n+1} &= \frac{B_{n,\,n+1}}{\alpha_{n+1}} c_{n+1,\,n+1} + \frac{B_{n,\,n-1}}{\alpha_{n-1}} c_{n-1,\,n+1} \\ \varepsilon_{n,\,n+3} &= \frac{B_{n,\,n+3}}{\alpha_{n+3}} + \frac{B_{n,\,n+1}}{\alpha_{n+1}} c_{n+1,\,n+3}. \end{aligned}$$

Finally, inserting the values of the α's, we have, as far as κ^2 upon the right-hand sides of the following,

$$\left.\begin{aligned}
\frac{\varepsilon_{n,n-3}}{2\kappa} &= -\frac{n-2}{6(n-1)(2n-3)}\gamma_{n,n-2} - \frac{1}{2(2n+1)}c_{n-1,n-3}\\
\frac{\varepsilon_{n,n-1}}{2\kappa} &= \frac{n-1}{2n(2n-3)}c_{n-1,n-1}\gamma_{n,n-2} - \frac{1}{2(2n+1)}\{c_{n-1,n-1}+c_{n+1,n-1}\}\\
\frac{\varepsilon_{n,n+1}}{2\kappa} &= \frac{n+2}{2(n+1)(2n+5)}c_{n+1,n+1}\gamma_{n,n+2} - \frac{1}{2(2n+1)}\{c_{n-1,n+1}+c_{n+1,n+1}\}\\
\frac{\varepsilon_{n,n+3}}{2\kappa} &= -\frac{n+3}{6(n+2)(2n+5)}\gamma_{n,n+2} - \frac{1}{2(2n+1)}c_{n+1,n+3},
\end{aligned}\right\} \quad . . \quad (73)$$

the c's and γ's being given by (70) and (71) respectively. The remaining ε's contain higher powers of κ than the second. The development of T_n by successive approximation, valid for any but small values of κ, is very laborious. We shall therefore require to consider its expansion by the method of series. Before doing this, however, we shall consider the expression for the divergent wave. Let H'_n be the second solution of equation (5), which has just been under consideration.

We have

$$\begin{aligned}
H'_n &= uv + w\\
&= \rho^{-\frac{1}{2}}R_n \cot^{-1}\frac{\rho}{\kappa} + \rho^{-\frac{1}{2}}T_n. \quad . . . . \quad (74)
\end{aligned}$$

Omitting an arbitrary constant factor the most general solution of (5) corresponding to n is

$$H_n + A_n H'_n,$$

where A_n is a constant, and

$$H_n = \rho^{-\frac{1}{2}}R_n.$$

Now, for large values of ρ,

$$F_s = J_{s+\frac{1}{2}}(\rho) = \left(\frac{2}{\pi\rho}\right)^{\frac{1}{2}}\cos(\rho - \tfrac{1}{2}\pi s - \tfrac{1}{2}\pi).$$

Hence

$$\begin{aligned}
F_n &= \left(\frac{2}{\pi\rho}\right)^{\frac{1}{2}}\cos\theta,\\
F_{n+2} &= -\left(\frac{2}{\pi\rho}\right)^{\frac{1}{2}}\cos\theta,\\
F_{n-2} &= -\left(\frac{2}{\pi\rho}\right)^{\frac{1}{2}}\cos\theta, \text{ etc.},\\
F_{n+1} &= \left(\frac{2}{\pi\rho}\right)^{\frac{1}{2}}\sin\theta.\\
F_{n-1} &= -\left(\frac{2}{\pi\rho}\right)^{\frac{1}{2}}\sin\theta, \text{ etc.}
\end{aligned}$$

where $\theta = \rho - \frac{1}{2}\pi n - \frac{1}{2}\pi$.

For large values of ρ

$$\cot^{-1}\frac{\rho}{\kappa} \to \frac{\kappa}{\rho}.$$

Hence, ultimately, when ρ is large

$$\mathrm{H}_n + \mathrm{A}_n\mathrm{H}'_n = \frac{1}{\rho}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \cos\theta \{\ldots + \gamma_{n,n-4} - \gamma_{n,n-2} + 1 - \gamma_{n,n+2} + \gamma_{n,n+4} - \ldots\}$$
$$+ \mathrm{A}_n \frac{1}{\rho}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin\theta \{\ldots + \varepsilon_{n,n-3} - \varepsilon_{n,n-1} + \varepsilon_{n,n+1} - \varepsilon_{n,n+3} + \ldots\}.$$

In order that this solution may represent a divergent wave it is necessary that

$$\mathrm{A}_n = -i\frac{\ldots + \gamma_{n,n-4} - \gamma_{n,n-2} + 1 - \gamma_{n,n+2} + \gamma_{n,n+4} - \ldots}{\ldots + \varepsilon_{n,n-3} - \varepsilon_{n,n-1} + \varepsilon_{n,n+1} - \varepsilon_{n,n+3} + \ldots} \quad . \quad . \quad . \quad . \quad (75)$$

Let us consider the denominator of this expression for A_n, which denominator will be denoted by D_n. We have without difficulty from (73)

$$\frac{\mathrm{D}_n}{2\kappa} = -\frac{2n-1}{2\,.\,3n\,(n-1)}\gamma_{n,n-2} + \frac{2n+3}{2\,.\,3\,(n+1)\,(n+2)}\gamma_{n,n+2}$$
$$+ \frac{1}{2\,(2n+1)}\{-c_{n-1,n-3} + c_{n-1,n-1} + c_{n+1,n-1} - c_{n-1,n+1} - c_{n+1,n+1} + c_{n+1,n+3}\}.$$

Putting in the values of the c's and γ's from (70) and (71) it is found that, when $n > 1$, the term independent of κ and the coefficient of κ^2 in $\mathrm{D}_n/2\kappa$ both vanish.

This result is of some importance, as will be seen later. It is only true if $n > 1$. The formulæ (73) must be used with caution when $n = 0$ or 1, but they should present no difficulty.

Another method of solving for T_n is the method of series, and to this we shall now proceed.

The Method of Series.

The equation for T_n is

$$\rho^2\frac{d^2\mathrm{T}_n}{d\rho^2} + \rho\frac{d\mathrm{T}_n}{d\rho} - (n+\tfrac{1}{2})^2\,\mathrm{T}_n + \rho^2\mathrm{T}_n + \frac{\kappa^2}{\rho^2}\left\{\rho^2\frac{d^2\mathrm{T}_n}{d\rho^2} - \rho\frac{d\mathrm{T}_n}{d\rho} + \tfrac{3}{4}\mathrm{T}_n\right\}$$
$$+ \kappa^2\mathrm{T}_n - a_n\mathrm{T}_n - 2\kappa\left\{\frac{d\mathrm{R}_n}{d\rho} - \frac{\mathrm{R}_n}{2\rho}\right\} = 0. \quad . \quad (76)$$

If the value of R_n be substituted in

$$2\kappa\left\{\frac{d\mathrm{R}_n}{d\rho} - \frac{\mathrm{R}_n}{2\rho}\right\}$$

we obtain

$$\ldots + \mathrm{B}_{n,n-3}\mathrm{F}_{n-3} + \mathrm{B}_{n,n-1}\mathrm{F}_{n-1} + \mathrm{B}_{n,n+1}\mathrm{F}_{n+1} + \mathrm{B}_{n,n+3}\mathrm{F}_{n+3} + \ldots,$$

where the B's are given by (64).

Assuming, as before, for the solution of (76)

$$T_n = \ldots + \varepsilon_{n,n-3}F_{n-3} + \varepsilon_{n,n-1}F_{n-1} + \varepsilon_{n,n+1}F_{n+1} + \varepsilon_{n,n+3}F_{n+3} + \ldots,$$

we substitute this expression in the differential equation. The result is

$$\begin{aligned}
&\varepsilon_{n,n-3}\{\alpha_{n-3}F_{n-3} - L_{n-3}F_{n-1} - M_{n-3}F_{n-3} - N_{n-3}F_{n-5}\}\\
&+ \varepsilon_{n,n-1}\{\alpha_{n-1}F_{n-1} - L_{n-1}F_{n+1} - M_{n-1}F_{n-1} - N_{n-1}F_{n-3}\}\\
&+ \varepsilon_{n,n+1}\{\alpha_{n+1}F_{n+1} - L_{n+1}F_{n+3} - M_{n+1}F_{n+1} - N_{n+1}F_{n-1}\}\\
&+ \varepsilon_{n,n+3}\{\alpha_{n+3}F_{n+3} - L_{n+3}F_{n+5} - M_{n+3}F_{n+3} - N_{n+3}F_{n+1}\}\\
&- B_{n,n-3}F_{n-3} - B_{n,n-1}F_{n-1} - B_{n,n+1}F_{n+1} - B_{n,n+3}F_{n+3}.
\end{aligned}$$

The coefficients in this result must vanish identically.

Let

$$M_s - \alpha_s = A_s,$$

as in (23).

Then

$$\begin{aligned}
&\varepsilon_{n,n-3}A_{n-3} + \varepsilon_{n,n-1}N_{n-1} + B_{n,n-3} = 0\\
&\varepsilon_{n,n-3}L_{n-3} + \varepsilon_{n,n-1}A_{n-1} + \varepsilon_{n,n+1}N_{n+1} + B_{n,n-1} = 0\\
&\varepsilon_{n,n-1}L_{n-1} + \varepsilon_{n,n+1}A_{n+1} + \varepsilon_{n,n+3}N_{n+3} + B_{n,n+1} = 0\\
&\varepsilon_{n,n+1}L_{n+1} + \varepsilon_{n,n+3}A_{n+3} + \varepsilon_{n,n+5}N_{n+5} + B_{n,n+3} = 0\\
&\varepsilon_{n,n+3}L_{n+3} + \varepsilon_{n,n+5}A_{n+5} + B_{n,n+5} = 0.
\end{aligned} \quad (77)$$

The terms omitted are

$$\varepsilon_{n,n-5}L_{n-5} \quad \text{and} \quad \varepsilon_{n,n+7}N_{n+7}$$

which are very small if κ be not too great. Even without them the equations are complicated, but we can take advantage of a further approximation.

When $n < 2$, $\varepsilon_{n,n-3} = 0$.

When $n > 1$, $\varepsilon_{n,n+5}N_{n+5}$ is very small and may be neglected.

To determine $\varepsilon_{n,n-3}$ the last equation of (77) may be omitted.

Let Δ' denote the determinant

$$\begin{vmatrix} A_{n-3} & N_{n-1} & 0 & 0 \\ L_{n-3} & A_{n-1} & N_{n+1} & 0 \\ 0 & L_{n-1} & A_{n+1} & N_{n+3} \\ 0 & 0 & L_{n+1} & A_{n+3} \end{vmatrix}$$

and let Δ'_{n-3} denote the determinant

$$\begin{vmatrix} B_{n,n-3} & N_{n-1} & 0 & 0 \\ B_{n,n-1} & A_{n-1} & N_{n+1} & 0 \\ B_{n,n+1} & L_{n-1} & A_{n+1} & N_{n+3} \\ B_{n,n+3} & 0 & L_{n+1} & A_{n+3} \end{vmatrix}$$

Then it is easily shown that

$$\varepsilon_{n,\,n-3} = -\frac{\Delta'_{n-3}}{\Delta'}.$$

Now let the minor of $B_{n,\,n-3}$ in the determinant Δ'_{n-3} be denoted by Δ, and the minor of N_{n-1} by Δ_{n-1}. Then

$$\varepsilon_{n,\,n-3} = \frac{-B_{n,\,n-3}\,\Delta + N_{n-1}\,\Delta_{n-1}}{A_{n-3}\,\Delta - N_{n-1}L_{n-3}\,\Delta_1},$$

where the determinant Δ_1 is obvious.

Now the product $N_{n-1}\,L_{n-3}$ is small. Hence, approximately,

$$A_{n-3}\varepsilon_{n,\,n-3} = -B_{n,\,n-3} + N_{n-1}\frac{\Delta_{n-1}}{\Delta}. \qquad (78)$$

Similarly

$$A_{n+5}\varepsilon_{n,\,n+5} = -B_{n,\,n+5} + L_{n+3}\frac{\Delta_{n+3}}{\Delta}. \qquad (79)$$

Having thus found approximations to $\varepsilon_{n,\,n-3}$ and $\varepsilon_{n,\,n+5}$, we can now determine $\varepsilon_{n,\,n-1}$, $\varepsilon_{n,\,n+1}$, and $\varepsilon_{n,\,n+3}$ from the three central equations of (77).

If, however, κ be not much greater than 3, we may neglect the products

$$\varepsilon_{n,\,n-3}L_{n-3} \quad \text{and} \quad \varepsilon_{n,\,n+5}N_{n+5}$$

in the determination of $\varepsilon_{n,\,n-1}$, $\varepsilon_{n,\,n+1}$, and $\varepsilon_{n,\,n+3}$.

Hence if, as before,

$$\Delta = \begin{vmatrix} A_{n-1} & N_{n+1} & 0 \\ L_{n-1} & A_{n+1} & N_{n+3} \\ 0 & L_{n+1} & A_{n+3} \end{vmatrix} \qquad \Delta_{n-1} = \begin{vmatrix} B_{n,\,n-1} & N_{n+1} & 0 \\ B_{n,\,n+1} & A_{n+1} & N_{n+3} \\ B_{n,\,n+3} & L_{n+1} & A_{n+3} \end{vmatrix}$$

and if further we write down the values of Δ_{n+3}, which has already been used, and of Δ_{n+1}, viz. :—

$$\Delta_{n+3} = \begin{vmatrix} A_{n-1} & N_{n+1} & B_{n,\,n-1} \\ L_{n-1} & A_{n+1} & B_{n,\,n+1} \\ 0 & L_{n+1} & B_{n,\,n+3} \end{vmatrix} \qquad \Delta_{n+1} = \begin{vmatrix} A_{n-1} & B_{n,\,n-1} & 0 \\ L_{n-1} & B_{n,\,n+1} & N_{n+3} \\ 0 & B_{n,\,n+3} & A_{n+3} \end{vmatrix}$$

it is not difficult to show that

$$\varepsilon_{n,\,n-1} = -\frac{\Delta_{n-1}}{\Delta}, \qquad (80)$$

$$\varepsilon_{n,\,n+1} = -\frac{\Delta_{n+1}}{\Delta}, \qquad (81)$$

$$\varepsilon_{n,\,n+3} = -\frac{\Delta_{n+3}}{\Delta}. \qquad (82)$$

The values of $\varepsilon_{n,\,n-3}$ and of $\varepsilon_{n,\,n+5}$ are then obtained from (78) and (79).

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We may repeat that κ is considered to be not much greater than 3. The approximations so far obtained are sufficient for all values of n. It is most convenient, however, to study the case, when $n = 0$, separately; for it is very much simpler than the other cases. Moreover, when $n > 0$, we may, without serious error, neglect the product $L_{n+1} N_{n+3}$.

Under these circumstances, which are in accordance with our general plan, we have without difficulty, when $n > 0$,

$$- \varepsilon_{n, n-1} = \frac{B_{n, n-1}A_{n+1} - B_{n, n+1}N_{n+1}}{A_{n-1}A_{n+1} - L_{n-1}N_{n+1}}, \quad \ldots \quad (83)$$

$$- \varepsilon_{n, n+1} = \frac{B_{n, n+1}A_{n-1} - B_{n, n-1}L_{n-1}}{A_{n-1}A_{n+1} - L_{n-1}N_{n+1}}, \quad \ldots \quad (84)$$

$$- \varepsilon_{n, n+3} = - \frac{B_{n, n+3}}{6(n+2)} - \frac{L_{n+1}}{6(n+2)} \varepsilon_{n, n+1}, \quad \ldots \quad (85)$$

$$- \varepsilon_{n, n-3} = \frac{B_{n, n-3}}{6(n-1)} + \frac{N_{n-1}}{6(n-1)} \varepsilon_{n, n-1}. \quad \ldots \quad (86)$$

In the equations (85) and (86) advantage has been taken of the fact that we may omit M_{n+3} and M_{n-3} respectively in the evaluation of A_{n+3} and A_{n-3} in comparison with α_{n+3} and α_{n-3}. When $n = 0$ the product $L_{n+1} N_{n+3}$ cannot reasonably be omitted. Hence, when $n = 0$,

$$- \varepsilon_{n, n+1} = \frac{B_{n, n+1}A_{n+3} - B_{n, n+3}N_{n+3}}{A_{n+1}A_{n+3} - L_{n+1}N_{n+3}}, \quad \ldots \quad (87)$$

$$- \varepsilon_{n, n+3} = \frac{B_{n, n+3}A_{n+1} - B_{n, n+1}L_{n+1}}{A_{n+1}A_{n+3} - L_{n+1}N_{n+3}}, \quad \ldots \quad (88)$$

$$- \varepsilon_{n, n+5} = - \frac{B_{n, n+5}}{10(n+3)} - \frac{L_{n+3}}{10(n+3)} \varepsilon_{n, n+3}. \quad \ldots \quad (89)$$

The approximation in (89) is obvious. The values of the B's are given by (64). For reference we shall write down the values of the A's, L's, and N's.

$$\left. \begin{aligned} A_{n-1} &= 2n + \frac{2n \cdot 2\kappa^2}{(2n+1)(2n-3)(2n+3)(2n-1)} - b_n \\ A_{n+1} &= -2(n+1) - \frac{2(n+1) \cdot 2\kappa^2}{(2n+5)(2n+1)(2n+3)(2n-1)} - b_n \\ A_{n+3} &= -6(n+2) - \frac{6(n+2) \cdot 2\kappa^2}{(2n+9)(2n+5)(2n+3)(2n-1)} - b_n \end{aligned} \right\} \quad \ldots \quad (90)$$

The most convenient expression for b_n appears to be the following :—

$$b_n = -\kappa^2 \frac{n(n-1)}{(2n-3)(2n-1)} \gamma_{n,\,n-2} - \kappa^2 \frac{(n+1)(n+2)}{(2n+5)(2n+3)} \gamma_{n,\,n+2}. \quad . \quad . \qquad (91)$$

Finally,

$$\left.\begin{aligned} L_{n-1} &= -\kappa^2 \frac{n(n+1)}{(2n-1)(2n+1)}, & N_{n-1} &= -\kappa^2 \frac{(n-2)(n-1)}{(2n-3)(2n-1)}, \\ L_{n+1} &= -\kappa^2 \frac{(n+2)(n+3)}{(2n+3)(2n+5)}, & N_{n+1} &= -\kappa^2 \frac{n(n+1)}{(2n+1)(2n+3)}, \\ L_{n+3} &= -\kappa^2 \frac{(n+4)(n+5)}{(2n+7)(2n+9)}, & N_{n+3} &= -\kappa^2 \frac{(n+2)(n+3)}{(2n+5)(2n+7)} \end{aligned}\right\} . \qquad (92)$$

The coefficients ε have now been obtained in terms of the coefficients γ. It will be found that in the calculation of the coefficients B, when $n > 1$, it is unnecessary to include any of the γ's except $\gamma_{n,\,n-2}$ and $\gamma_{n,\,n+2}$.

When $n = 0$ it is advisable to retain both γ_{02} and γ_{04} in the coefficients ε, and when $n = 1$ both γ_{13} and γ_{15}. The effect of γ_{04} and also of γ_{15} is very small, but their inclusion will serve as a check upon our approximations.

For the purpose of the important applications which we have in view the formulæ for the coefficients γ, which are sufficiently accurate if κ be not much greater than 3 and if n have any integral value including zero, may here be tabulated.

When $n = 0$ and 1 the terms containing κ^6 are included.

When $n = 0$.

$$\left.\begin{aligned} \gamma_{02} &= -\beta_{02} = -\left(\frac{\kappa}{3}\right)^2 \left\{1 + \frac{2}{7}\cdot\left(\frac{\kappa}{3}\right)^2 - \frac{13}{7\,.\,25}\left(\frac{\kappa}{3}\right)^4\right\} \\ \gamma_{04} &= \beta_{04} = \frac{9\,.\,3}{25\,.\,7}\left(\frac{\kappa}{3}\right)^4 \left\{1 + \frac{4}{11}\left(\frac{\kappa}{3}\right)^2\right\} \\ \gamma_{06} &= -\beta_{06} = -\frac{27}{5\,.\,7\,.\,11\,.\,7}\left(\frac{\kappa}{3}\right)^6 \end{aligned}\right\} . \quad . \quad . \quad . \quad . \qquad (93)$$

When $n = 1$.

$$\left.\begin{aligned} \gamma_{13} &= -\beta_{13} = -\frac{9}{25}\left(\frac{\kappa}{3}\right)^2 \left\{1 - \frac{2}{25}\left(\frac{\kappa}{3}\right)^2 - \frac{229\,.\,9}{5\,.\,7\,.\,25\,.\,77}\left(\frac{\kappa}{3}\right)^4\right\} \\ \gamma_{15} &= \beta_{15} = \frac{9}{5\,.\,49}\left(\frac{\kappa}{3}\right)^4 \left\{1 - \frac{4\,.\,9}{13\,.\,25}\left(\frac{\kappa}{3}\right)^2\right\} \\ \gamma_{17} &= -\beta_{17} = -\frac{9}{5\,.\,7\,.\,11\,.\,13}\cdot\left(\frac{\kappa}{3}\right)^6 \end{aligned}\right\} . \quad . \qquad (93\text{A})$$

When n is greater than 1 the general formulæ are

$$\left.\begin{aligned}
\gamma_{n,n-4} &= \kappa^4 \frac{n(n-1)(n-2)(n-3)}{(2n+1)(2n-1)^2(2n-3)^2(2n-5)\,2\,.\,4} \\
\gamma_{n,n-2} &= \kappa^2 \frac{n(n-1)}{(2n+1)(2n-1)^2\,2}\left\{1 - \frac{2\kappa^2}{(2n-1)^2(2n-5)(2n+3)}\right\}, \\
\gamma_{n,n+2} &= -\kappa^2 \frac{(n+1)(n+2)}{(2n+1)(2n+3)^2\,2}\left\{1 - \frac{2\kappa^2}{(2n+3)^2(2n+7)(2n-1)}\right\}, \\
\gamma_{n,n+4} &= \kappa^4 \frac{(n+1)(n+2)(n+3)(n+4)}{(2n+1)(2n+3)^2(2n+5)^2(2n+7)\,2\,.\,4}
\end{aligned}\right\} . \quad (94)$$

The Solutions H_n and H'_n.

Two independent solutions of equation (5) have now been obtained. These solutions have been denoted respectively by H_n and H'_n. They are both convergent when the variable ρ is zero, and they both vanish as $1/\rho$ when ρ becomes very great.

The general solution of (5) may be obtained by a combination of H_n and H'_n, and if this solution is required to represent a divergent wave it must, as has been seen, be of the form

$$H_n + A_n H'_n,$$

where A_n is given by (75).

We shall denote this solution by ψ_n, so that

$$\psi_n = H_n + A_n H'_n. \quad (95)$$

The constant A_n is conveniently denoted by $-iC_n/D_n$, where

$$C_n = \ldots + \gamma_{n,n-4} - \gamma_{n,n-2} + 1 - \gamma_{n,n+2} + \gamma_{n,n+4} - \ldots$$

and where, as before,

$$D_n = \ldots + \epsilon_{n,n-3} - \epsilon_{n,n-1} + \epsilon_{n,n+1} - \epsilon_{n,n+3} + \ldots \quad (96)$$

Properties of the Functions ψ_n.

The properties of the function ψ_n, when ρ is small, are important. Written in full

$$\psi_n = \rho^{-\frac{1}{2}} R_n + A_n \rho^{-\frac{1}{2}} \left(R_n \cot^{-1}\frac{\rho}{\kappa} + T_n\right).$$

When ρ is very small

$$\cot^{-1}\rho/\kappa = \tfrac{1}{2}\pi$$

and

$$\frac{d}{d\rho}(\cot^{-1}\rho/\kappa) = -\frac{1}{\kappa}.$$

Also, when ρ is very small,

$$\rho^{-\frac{1}{2}} F_n(\rho) = \rho^{-\frac{1}{2}} J_{n+\frac{1}{2}}(\rho) = 0,$$

except when $n = 0$, and then

$$\rho^{-\frac{1}{2}} F_n(\rho) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}.$$

Again

$$\frac{d}{d\rho}\{\rho^{-\frac{1}{2}} F_n(\rho)\} = 0,$$

except when $n = 1$, and then

$$\frac{d}{d\rho}\{\rho^{-\frac{1}{2}} F_n(\rho)\} = \tfrac{1}{3}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}.$$

The values of ψ_n and of $d\psi_n/d\rho$, when ρ is very small, can now be written down without difficulty.

When n is even, and $\rho = 0$,

$$\left.\begin{aligned} \psi_n &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}}(1 + \tfrac{1}{2}\pi \,.\, A_n)\,\gamma_{n0} \\ \frac{d\psi_n}{d\rho} &= -\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{1}{\kappa}\,A_n\gamma_{n0} + \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\tfrac{1}{3}A_n\varepsilon_{n1}. \\ &\text{When } n \text{ is odd, and } \rho = 0, \\ \psi_n &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} A_n\varepsilon_{n0} \\ \frac{d\psi_n}{d\rho} &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\tfrac{1}{3}\,(1 + \tfrac{1}{2}\,\pi\, A_n)\,\gamma_{n1} \end{aligned}\right\} \quad \ldots\ldots \quad (97)$$

Equally important are the values of the functions when ρ is very large.

When ρ is large

$$\left.\begin{aligned} H_n &\to C_n\,\frac{1}{\rho}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\cos(\rho - \tfrac{1}{2}\pi n - \tfrac{1}{2}\pi), \\ H'_n &\to D_n\,\frac{1}{\rho}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\sin(\rho - \tfrac{1}{2}\pi n - \tfrac{1}{2}\pi), \\ \psi_n &\to C_n\,\frac{1}{\rho}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} e^{-i(\rho - \frac{1}{2}\pi n - \frac{1}{2}\pi)} \end{aligned}\right\} \quad \ldots\ldots \quad (98)$$

The Boundary Conditions.

In problems of fluid motion the condition satisfied at the boundary of the fluid has to be expressed analytically.

The surfaces bounding the fluid are either $\beta = \text{const.}$, or $\mu = \text{const.}$

If no fluid enters an elementary ring of the surface $\beta = \text{const.}$, then

$$\frac{\partial \phi}{\partial s_\beta} \delta s_\mu \,.\, 2\pi r = 0,$$

where ϕ is the velocity potential.

Now

$$\frac{\partial \beta}{\partial s_\beta} \,.\, \delta s_\mu \,.\, 2\pi r = 2\pi a\,(\beta^2 + 1)\,\delta\mu.$$

Hence the condition is that

$$\frac{\partial \phi}{\partial \beta} \,.\, 2\pi a\,(\beta^2 + 1)\,\delta\mu = 0.$$

The boundary condition is, therefore, that $\partial\phi/\partial\beta$ vanishes, which is equivalent to the vanishing of $\partial\phi/\partial\rho$.

Again, if no fluid enters an elementary ring of the surface $\mu = \text{const.}$, then

$$\frac{\partial \phi}{\partial \mu} \,.\, 2\pi a\,(1 - \mu^2)\,\delta\beta = 0.$$

The boundary condition in this case is that $\partial\phi/\partial\mu$ vanishes.

The boundary conditions for the waves incident from a Hertzian Oscillator with its axis lying along the axis of symmetry are of the same form.

The Function D_n.

In applications the most troublesome quantity with which we have to deal is that which has been denoted by D_n. It is therefore of advantage to discuss this quantity in a little more detail before proceeding to specified problems.

In the expressions (84) for $-\varepsilon_{n,\,n+1}$ and (83) for $-\varepsilon_{n,\,n-1}$ the common denominator can be simplified without serious loss of accuracy if n be greater than 1 and if κ be restricted to values not much greater than 3.

The term $L_{n-1}\,N_{n+1}$ in the denominator may, under these circumstances, be neglected.

Thus for $-\varepsilon_{n,\,n+1}$ we may write

$$\frac{B_{n,\,n+1}}{A_{n+1}} - \frac{B_{n,\,n-1}\,L_{n-1}}{A_{n+1}\,A_{n-1}}.$$

Again, without serious loss of accuracy under the same circumstances, A_{n+1} and A_{n-1} may be replaced by their first terms. Hence

$$-\varepsilon_{n,\,n+1} = -\frac{B_{n,\,n+1}}{2\,(n+1)} + \frac{B_{n,\,n-1}L_{n-1}}{2n\,.\,2\,(n+1)}.$$

Recalling that in the evaluation of D_n, when $n > 1$, the terms containing κ and κ^3 vanish, it is easy to show that the contribution of the second term to D_n is small. Thus a first approximation for the contribution of $-\varepsilon_{n, n+1}$ to D_n is contained in

$$-\frac{B_{n, n+1}}{2(n+1)}$$

Similarly, without serious loss of accuracy,

$$-\varepsilon_{n, n-1} = \frac{B_{n, n-1}}{2n} + \frac{B_{n, n+1} N_{n+1}}{2n \, . \, 2(n+1)},$$

and a first approximation for the contribution of $-\varepsilon_{n, n-1}$ to D_n is contained in

$$\frac{B_{n, n-1}}{2n}.$$

Hence

$$\begin{aligned} D_n &= \ldots + \varepsilon_{n, n-3} - \varepsilon_{n, n-1} + \varepsilon_{n, n+1} - \varepsilon_{n, n+3} + \ldots . \\ &= -\frac{B_{n, n-3}}{6(n-1)} + \frac{N_{n-1}}{6(n-1)} \cdot \frac{B_{n, n-1}}{2n} + \frac{B_{n, n-1}}{2n} + \frac{N_{n+1}}{2n} \frac{B_{n, n+1}}{2(n+1)} \\ &\quad + \frac{B_{n, n+1}}{2(n+1)} - \frac{L_{n-1}}{2(n+1)} \cdot \frac{B_{n, n-1}}{2n} - \frac{B_{n, n+3}}{6(n+2)} - \frac{L_{n+1}}{6(n+2)} \cdot \frac{B_{n, n+1}}{2(n+1)}, \end{aligned}$$

from which two terms in $\varepsilon_{n, n-3}$ and $\varepsilon_{n, n+3}$ respectively, that from the foregoing considerations are negligible, have been omitted.

The coefficients B are given by (64) in terms of the coefficients γ. Now the terms containing κ^2 in $\gamma_{n, n-2}$ and in $\gamma_{n, n+2}$ need only be retained, and $\gamma_{n, n-4}$ and $\gamma_{n, n+4}$ may be omitted, when $n > 1$. Again recalling that the coefficients of κ and of κ^3 in D_n both vanish, it follows that

$$\begin{aligned} D_n = &-\frac{2\kappa}{4n} \left\{ \frac{N_{n-1}}{3(n-1)} - \frac{L_{n-1}}{n+1} \right\} \frac{n-1}{2n-3} \gamma_{n, n-2} \\ &+ \frac{2\kappa}{4(n+1)} \left\{ \frac{N_{n+1}}{n} - \frac{L_{n+1}}{3(n+2)} \right\} \frac{n+2}{2n+5} \gamma_{n, n+2}. \end{aligned}$$

Hence, upon insertion of the values of N_{n-1}, N_{n+1}, L_{n-1}, and L_{n+1}, it follows that

$$D_n = -\frac{\kappa^3}{3(2n+1)} \left\{ \frac{1}{n} \left(\frac{n-1}{2n-3} \right)^2 \gamma_{n, n-2} + \frac{1}{n+1} \left(\frac{n+2}{2n+5} \right)^2 \gamma_{n, n+2} \right\} . \quad . \quad . \quad (99)$$

When $n = 0$ or 1 the approximations, which have been found to be valid in the evaluation of D_n when $n > 1$, are no longer permissible.

To these cases we must now give more attention.

When $n = 1$ denote the common denominator of the expressions for $-\varepsilon_{n, n-1}$ and $-\varepsilon_{n, n+1}$ in (83) and (84) by Δ. Note also that $\varepsilon_{n, n-3}$ vanishes. Then

$$\Delta \,.\, D_n = B_{n, n-1} A_{n+1} - B_{n, n+1} N_{n+1} - B_{n, n+1} A_{n-1} + B_{n, n-1} L_{n-1} - \frac{B_{n, n+3}}{6(n+2)} \Delta + \frac{L_{n+1}}{6(n+2)} \{B_{n, n+1} A_{n-1} - B_{n, n-1} L_{n-1}\} . \quad . \quad . \quad (100)$$

When $n = 0$, $B_{n, n+5}$ may be omitted without serious error, and, if the common denominator of the expressions for $-\varepsilon_{n, n+1}$ and $-\varepsilon_{n, n+3}$ in (87) and (88) be denoted by Δ,

$$\Delta \,.\, D_n = -B_{n, n+1} A_{n+3} + B_{n, n+3} N_{n+3} + (B_{n, n+3} A_{n+1} - B_{n, n+1} L_{n+1}) \left\{1 - \frac{L_{n+3}}{10(n+3)}\right\} . \quad . \quad . \quad . \quad (101)$$

Physical Applications.

The functions necessary for the most important physical applications have now been developed. The analysis could be completed and the application extended by the development of a second solution of equation (4), analogous to the second solution of equation (5) which has been denoted by H'_n. We shall merely designate this second solution of equation (4) by E'_n, for it will not actually be required in its formal development.

The general problems that can be dealt with can be placed under two headings. In both, the surface bounding space externally is, if present, one of the ellipsoidal surfaces. But this surface may be absent.

Under one heading the surface bounding space internally is one of the confocal ellipsoidal surfaces.

Under the other heading space is bounded internally by one of the confocal hyperboloidal surfaces.

Scattering by a Thin Circular Disc.

The first problem to be considered is the normal incidence of plane waves of sound upon a thin circular disc and their subsequent scattering. The expression for the incident plane wave is

$$I_0 = e^{i\kappa(z+ct)} = I e^{i\kappa ct}.$$

Now

$$i\kappa z = i\kappa_1 a \sinh \alpha \,.\, \mu = i\kappa\beta\mu = i\rho\mu.$$

Hence

$$I = e^{i\rho\mu}.$$

When ρ is small

$$I = 1 + i\rho\mu,$$

which represents the wave, therefore, in the neighbourhood of the thin disc. For μ substitute its expansion in the ellipsoidal functions given by (19). Then, in the neighbourhood of the disc,

$$I = 1 + i\rho\,\tfrac{2}{3}\left(\frac{1}{e_1}E_1 + \frac{\beta_{31}}{e_3}E_3 + \frac{\beta_{51}}{e_5}E_5 + \ldots\right). \quad \quad (102)$$

The subsequent terms in the expansion are quite negligible.

The scattered wave must be expressed in terms of the ψ functions, which are suitable for divergent wave motion.

Denote, therefore, the scattered wave by S_0 where

$$S_0 = Se^{i\kappa ct},$$

and assume for S the expansion

$$S = a(a_1\,\psi_1\,E_1 + a_3\,\psi_3\,E_3 + a_5\,\psi_5\,E_5). \quad \quad (103)$$

The boundary condition requires that

$$\frac{d}{d\rho}(I + S)$$

vanishes identically.

Hence, at the surface of the disc,

$$i\tfrac{2}{3}\cdot\frac{1}{e_1} + aa_1\frac{d\psi_1}{d\rho} = 0$$

$$i\tfrac{2}{3}\frac{\beta_{31}}{e_3} + aa_3\frac{d\psi_3}{d\rho} = 0, \text{ etc.}$$

The values of $d\psi_n/d\rho$ when ρ is very small are given by (97). Hence

$$i\tfrac{2}{3}\frac{1}{e_1} + aa_1\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\tfrac{1}{3}(1 + \tfrac{1}{2}\pi A_1) = 0$$

$$i\tfrac{2}{3}\frac{\beta_{31}}{e_3} + aa_3\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\tfrac{1}{3}(1 + \tfrac{1}{2}\pi A_3)\,\gamma_{31} = 0, \text{ etc.}$$

Remembering that

$$\beta_{31} = -\gamma_{31}$$

$$\beta_{51} = \gamma_{51}, \text{ etc.},$$

and putting $a = i\left(\frac{\pi}{2}\right)^{\frac{1}{2}}$, we obtain

$$a_1 = -\frac{2}{e_1}\frac{1}{1 + \frac{1}{2}\pi A_1}$$

$$a_3 = +\frac{2}{e_3}\frac{1}{1 + \frac{1}{2}\pi A_3}, \text{ etc.}$$

The most interesting cases of the motion are those close to the disc and those at a great distance from the disc. When ρ is small the values of ψ_n are given by (97). In this case

$$S = a\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\{a_1 A_1 \varepsilon_{10} E_1 + a_3 A_3 \varepsilon_{30} E_3 + a_5 A_5 \varepsilon_{50} E_5\}. \quad . \quad . \quad . \quad . \quad . \quad (104)$$

When ρ is large we have from (98)

$$S = a\frac{i}{\rho}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} e^{-i(\rho-\frac{1}{2}\pi)}\{a_1 C_1 E_1 - a_3 C_3 E_3 + a_5 C_5 E_5\}. \quad . \quad . \quad . \quad . \quad . \quad (105)$$

We have, therefore, to consider, when ρ is small, the coefficient $a_n A_n \varepsilon_{n0}$.
Since n is odd

$$a_n A_n \varepsilon_{n0} = i^n \frac{2}{e_n} \cdot \frac{C_n}{D_n - \frac{1}{2}\pi i C_n} \varepsilon_{n0}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (106)$$

where e_n is given by (17) and C_n and D_n by (96).

When ρ is large we have to consider the coefficient $a_n C_n$, obtained from the formula

$$a_n C_n = i^{n+1} \frac{2}{e_n} \cdot \frac{C_n D_n}{D_n - \frac{1}{2}\pi i C_n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (107)$$

The component of the scattered wave, which, when ρ is small, involves $a_5 A_5 \varepsilon_{50}$, may be neglected, for ε_{50} is extremely small even when $\kappa = 3$.

The same component of the scattered wave, which, when ρ is large, involves $a_5 C_5$, may again be neglected, for D_5 is extremely small even when $\kappa = 3$. Thus the scattered wave consists effectively of two components corresponding to $n = 1$ and $n = 3$ respectively.

We shall consider first the component corresponding to $n = 1$. Since this component is very complicated it is convenient to write down separately the values of

$$e_1 C_1 D_1 \text{ and } \varepsilon_{1,0}.$$

From (17) we obtain with sufficient accuracy

$$e_1 = \tfrac{2}{3}(1 + \tfrac{3}{7} \cdot \gamma^2{}_{13}).$$

From (96)

$$C_1 = 1 - \gamma_{13} + \gamma_{15}$$

$$= 1 + \frac{1}{25}\kappa^2 + \frac{3}{25 \cdot 25 \cdot 49}\kappa^4,$$

as far as the fourth power of κ.

With ample accuracy, therefore,

$$C_1 = 1 + \tfrac{1}{25}\kappa^2.$$

In the expression (100) for $\Delta \cdot D_1$ let us consider the factor Δ.

With sufficient accuracy

$$\Delta = -\left(2 - \frac{2 \cdot 2}{3 \cdot 5}\kappa^2 - \frac{2 \cdot 3}{7 \cdot 5 \cdot 25}\kappa^4\right)\left(4 + \frac{2 \cdot 2 \cdot 2}{3 \cdot 5 \cdot 7}\kappa^2 + \frac{2 \cdot 3}{7 \cdot 5 \cdot 25}\kappa^4\right) - \frac{2 \cdot 2}{3 \cdot 3 \cdot 5}\kappa^4.$$

The importance of retaining this somewhat complicated expression for Δ is now clear. When κ is small Δ has the value -8, but the absolute value of Δ becomes smaller and smaller as κ increases until, for a value of κ^2 equal to 12 very nearly, Δ vanishes. Thereafter Δ becomes positive. The approximations to which we have been working are valid with moderate accuracy for a value of κ equal to 3·5, that is for a value of κ^2 equal to 12.25, so that the same difficulty would occur at this value of κ in an attempt to expand the coefficients ε in ascending powers of κ as was encountered in the determination of b_n. Consider now the value of $\Delta \, . \, D_1$. The term

$$-\frac{B_{n,\,n+3}}{6\,(n+2)}\,\Delta = \frac{4 \cdot 2\kappa}{7 \cdot 18}\,\gamma_{13} \, . \, \Delta \text{ approximately,}$$

and this, as will be seen from (64), is a sufficient approximation. In our range it is always very small. The coefficient of $-B_{n,\,n+1}$ is, after a little reduction by using the equations (90), (91), and (92),

$$2 - \frac{8}{21}\kappa^2 - \frac{8}{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7}\kappa^2 - b_1.$$

The terms containing κ^6 may be omitted as being relatively small.

The coefficient of $B_{n,\,n-1}$ is

$$-4 - \frac{26}{35}\kappa^2 - \frac{8}{5 \cdot 7}\kappa^4 - b_1,$$

the terms containing κ^6 being likewise omitted.

Inserting the values of $B_{n,\,n+1}$ and $B_{n,\,n-1}$ from (64), neglecting the term containing $B_{n,\,n+3}$, and rearranging, we have with ample accuracy

$$\frac{\Delta \, . \, D_1}{2\kappa} = -\frac{2 \cdot 368}{5 \cdot 9 \cdot 35}\kappa^2 - \frac{8 \cdot 47}{3 \cdot 5 \cdot 7 \cdot 45}\kappa^4 + \frac{3}{7}\gamma_{13}\left\{\frac{82}{5 \cdot 21}\kappa^2 + \frac{8}{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7}\kappa^4 - \frac{2 \cdot 3}{7 \cdot 5}\kappa^2\gamma_{13}\right\}.$$

Since γ_{13} is negative it will be noted that every term upon the right-hand side of this equation is negative.

If the term containing $B_{n,\,n+3}$ be retained it is found that, as far as the second power of κ,

$$\frac{D_1}{2\kappa} = \left\{\frac{368}{5 \cdot 9 \cdot 35 \cdot 4} - \frac{8}{7 \cdot 18 \cdot 25 \cdot 2}\right\}\kappa^2$$
$$= \tfrac{2}{35}\kappa^2. \qquad (108)$$

This expression is accurate as far as the second power of κ, as may be seen by retracing the approximations used. Finally, when $n = 1$, we have

$$-\Delta \,.\, \varepsilon_{1,0} = B_{n,n-1} A_{n+1} - B_{n,n+1} N_{n+1}.$$

After a little reduction this becomes

$$\frac{\Delta}{2\kappa}\,\varepsilon_{1,0} = \tfrac{4}{3} + \tfrac{4}{7}\kappa^2 - \frac{4}{5\,.\,7}\,\kappa^2 \cdot \gamma_{13}. \qquad \text{(109)}$$

When $n = 3$ the values of

$$e_3\ C_3\ D_3 \text{ and } \varepsilon_{3,0}$$

are somewhat simpler.

From (17)

$$\varepsilon_3 = \tfrac{2}{7}\left(\tfrac{7}{3}\gamma_{31}^2 + 1 + \tfrac{7}{11}\gamma_{35}^2\right),$$

the other terms being quite negligible. With little error in fact

$$e_3 = \tfrac{2}{7}.$$

$$C_3 = -\gamma_{31} + 1 - \gamma_{35},$$

and we find easily, since the terms containing higher powers of κ than the second are quite negligible, that

$$C_3 = 1 + \frac{\kappa^2}{25\,.\,81}\,.$$

With sufficient accuracy we may put

$$C_3 = 1.$$

From (99) it is found readily that, with little error

$$D_3 = \left(\tfrac{1}{3}\kappa\right)^5 \cdot \tfrac{1}{50}.$$

Finally, when $n = 3$,

$$-\varepsilon_{3,0} = \frac{B_{n,n-3}}{6\,(n-1)} - \frac{N_{n-1}}{6\,(n-1)} \cdot \frac{B_{n,n-1}}{2n}\,.$$

In this expression powers of κ higher than the third may be neglected and we have, after a little reduction,

$$\varepsilon_{3,0} = -\frac{4}{5\,.\,5\,.\,7\,.\,9}\,\kappa^3.$$

Case I. κ small.—In the first place let ρ be small and consider the motion close to the disc. In this case

$$S = i a_1 A_1 \varepsilon_{10} E_1$$

$$= i\,\frac{2}{\pi}\,\kappa\mu.$$

Let $d\sigma$ be an element of area upon the disc, and let us integrate S over one face of the disc. Then

$$\iint S\,d\sigma = -i\,\frac{2}{\pi}\,\kappa\int_1^0 2\pi a^2\mu^2\,d\mu$$

$$= i\kappa_1\,\frac{4a^3}{3},$$

where $\kappa_1 = 2\pi/\lambda$.

This agrees with the expression for the diffraction of long waves.* There it will be found that

$$\iint S\,d\sigma = \tfrac{1}{2}i\kappa_1 M,$$

where

$$M = \tfrac{8}{3}a^3.$$

M is called the "inertia coefficient" of the disc.

When ρ is large

$$S = -\frac{a}{\rho}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} e^{-i\rho} a_1 C_1 E_1$$

$$= -\frac{1}{\rho}\,e^{-i\rho}\,3\,.\,\frac{2}{\pi}\,D_1\mu.$$

Now, when κ is small, from (108)

$$D_1 = \tfrac{4}{35}\kappa^3.$$

Also, when ρ is large,

$$\rho = \kappa_1\,a\sinh\alpha$$

$$= \kappa_1\,R,$$

where R is the radius vector drawn from the centre of the disc.

Hence

$$S = -\tfrac{36}{35}\,.\,\tfrac{8}{3}\,\frac{e^{-i\kappa_1 R}}{R}\,.\,\frac{\pi a^3}{\lambda^2}\,.\,\mu.$$

This expression differs slightly, to be precise by one thirty-fifth, from that obtained by the theory of the diffraction of long waves.†

The discrepancy in the latter theory is to be expected from considerations of the periodic solution of the fundamental differential equation.

There is a difficulty in the latter theory which requires consideration. The motion at every instant close to the disc is assumed to be that of an incompressible fluid. Now suppose the disc to be oscillating, and let it change its velocity in a small time Δt. For the foregoing assumption to be valid it is necessary that the diameter of the disc should be small in comparison with $c\Delta t$, where c is the velocity of wave propagation. If λ be the wave-length and T the periodic time

$$c\,\Delta t = \frac{\lambda}{T}\,\Delta t.$$

* See LAMB, "Hydrodynamics," p. 511, 4th Ed.

† LAMB, *loc. cit.*, p. 512.

If r be the radius of a sphere whose centre is at the centre of the disc, and if $r = c\Delta t$, then the diameter of the disc must be small compared with r, and r must be small compared with λ.

The change in energy of the motion of the disc in time Δt is merely communicated to the fluid within the sphere of radius r, as a first approximation, that is when the first power of κ only is retained in the expression for the motion close to the disc. Radiation depends upon the higher approximations, and this is verified clearly in the theory here developed by the fact that the first power of κ disappears in the expressions for the scattered wave.

Case II. *κ in the neighbourhood of* 3.—When κ is large but not much greater than 3, some very interesting results can be obtained from the theory so far developed. In fact, when $\kappa = \pi$ the diameter of the disc is equal to a wave-length, and the development of an approximate theory is fully justified if it can be applied to this problem. In the first place let ρ be small and consider the motion close to the disc. In this case

$$S = i\,\{a_1 A_1 \varepsilon_{10} E_1 + a_3 A_3 \varepsilon_{30} E_3\},$$

where, from (106),

$$a_n A_n \varepsilon_{n0} = i^n \frac{2C_n}{e_n} \cdot \frac{D_n + \frac{1}{2}\pi i C_n}{D_n^2 + \frac{1}{4}\pi^2 C_n^2}\, \varepsilon_{n0}.$$

These two terms, which we have taken to represent S, are amply sufficient if κ does not assume values much greater than 3. For values of κ as great as this, however, the discussion becomes very laborious.

We make use of the remarkable variations of D_n. On account of its importance a graph (fig. 1) is appended of D_1 for values of κ^2 up to 12. This graph is reasonably

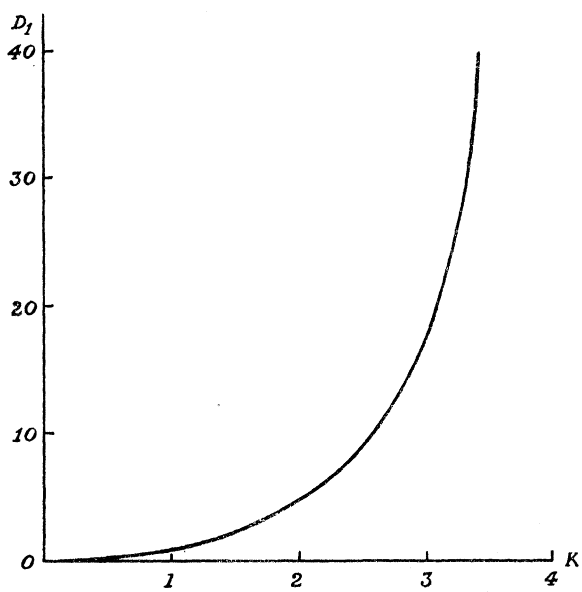

Fig. 1.

accurate if κ does not exceed 3·25, but for greater values of κ it is unreliable. The value of D_3 is obtained from (99). When κ has the value 3 we find from the expressions for $\Delta . D_1$ and Δ that, very approximately, $D_1 = 18$. Also for this value of κ the approximate value of D_3 is $\frac{1}{50}$.

We can say, then, that in the neighbourhood of the value $\kappa = 3$, or to be more precise for the range of values of κ lying between 2·75 and 3·25, say, C_1 is small in comparison with D_1 and C_3 is large in comparison with D_3. These considerations effect a great simplification. In fact, writing $S = S_1 + iS_2$, the approximation to S becomes

$$S = S_1 + iS_2$$
$$= -\frac{2C_1}{e_1 D_1}\,\varepsilon_{10} E_1 + i\,\frac{4}{\pi e_3}\,\varepsilon_{30} E_3. \qquad (110)$$

Now in the small range of values of κ under consideration C_1, e_1, and e_3 change very slowly, so that their values when $\kappa = 3$ are amply sufficient.

When $\kappa = 3$, $2C_1/e_1 = 4$, approximately, and $e_3 = 2/7$. Also

$$\varepsilon_{30} = -\frac{4\kappa^3}{5\,.\,5\,.\,7\,.\,9}.$$

It remains, therefore, to consider the factor $-\varepsilon_{10}/D_1$.

Both numerator and denominator of this factor, as far as our approximations require us to take them, consist of a series of ascending powers of κ^2, each term of which is positive.

We may write, then,

$$-\frac{\varepsilon_{10}}{D_1} = \frac{1}{\kappa^2}\cdot\frac{20}{3}\left\{\frac{7 + \text{powers of } \kappa^2}{16 + \text{powers of } \kappa^2}\right\}.$$

In the range of values of κ which we are considering the term within brackets changes very slowly with κ. The value of this term, when $\kappa = 3$, is very approximately $\frac{3}{4}$. Hence with quite tolerable accuracy we may say that, within this range,

$$-\frac{\varepsilon_{10}}{D_1} = \frac{5}{\kappa^2}.$$

Inserting the approximate value of $2C_1/e_1$, it follows that

$$S_1 = \frac{20}{\kappa^2}\,E_1.$$

The approximate value of S_2 is

$$S_2 = -\frac{1}{\pi}\,\frac{4\,.\,2}{5\,.\,5\,.\,9}\,\kappa^3 E_3.$$

Now

$$E_1 = P_1 + \beta_{13} P_3 + \beta_{15} P_5$$
$$= P_1 + \left(\frac{1}{25}\kappa^2 - \frac{2}{9\,.\,25\,.\,25}\kappa^4\right) P_3 + \frac{1}{5\,.\,9\,.\,49}\kappa^4 P_5,$$

as far as κ^4.

Without very serious error, therefore, we may write, even when κ is as great as 3·25,

$$E_1 = P_1 + \frac{\kappa^2}{25} P_3.$$

Again

$$E_3 = \beta_{31}P_1 + P_3 + \beta_{35}P_5,$$

$$= -\frac{3}{7 \cdot 25}\kappa^2 P_1 + P_3 + \frac{10}{7 \cdot 81}\kappa^2 P_5,$$

as far as κ^2.

We shall now consider the meaning of the expressions found.

For the total velocity potential near the disc we must add the incident wave I, which, in this neighbourhood, has the value unity. Hence, near the disc, the velocity potential

$$I + S = 1 + S_1 + iS_2.$$

This expression leads to the variable part of the pressure at the surface of the disc.

The velocity of the fluid close to the disc is expressed by

$$-\frac{d}{ds_\mu}(I + S), \quad \ldots \ldots \ldots \quad (111)$$

and upon the square of this expression depends the mean pressure at the surface of the disc.

Now, close to the disc

$$\frac{d\mu}{ds_\mu} = \frac{(1-\mu^2)^{\frac{1}{2}}}{a\mu}.$$

Hence, upon substituting for P_1 and P_3 their values in terms of μ, the real part of (111) is found to be

$$-\frac{dS_1}{ds_\mu} = -\left(6\mu^2 + \frac{20}{\kappa^2} - \tfrac{6}{5}\right)(1-\mu^2)^{\frac{1}{2}}\frac{1}{a\mu}. \quad \ldots \ldots \quad (112)$$

With regard to $-dS_2/dS\mu$ this is found to be relatively small over a considerable area of the disc, so that it may be neglected, so far as the amplitude is concerned, over this area.

The velocity in the incident wave being $-i\kappa_1$ it is of interest to consider the ratio

$$\frac{1}{\kappa_1}\frac{dS_1}{ds_\mu}. \quad \ldots \ldots \ldots \quad (113)$$

If F denote this ratio

$$F = \frac{1}{\kappa}\left(6\mu^2 + \frac{20}{\kappa^2} - \tfrac{6}{5}\right)\frac{(1-\mu^2)^{\frac{1}{2}}}{\mu}.$$

Let us consider, for example, the case of $\kappa = 3\cdot25$. In this case

$$\kappa F = (6\mu^2 + \tfrac{2}{3}) \frac{(1-\mu^2)^{\frac{1}{2}}}{\mu}$$

very approximately.

Hence

$$\kappa F = (6\cos^2\theta + \tfrac{2}{3}) \tan\theta.$$

From this

$$\frac{d}{d\theta}(\kappa F) = \frac{12}{\mu^2}(\mu^4 - \tfrac{1}{2}\mu^2 + \tfrac{1}{18}).$$

Therefore κF has a maximum when $\mu^2 = \frac{1}{3}$ and a minimum when $\mu^2 = \frac{1}{6}$. There is a point of inflexion when $\mu^2 = \dfrac{1}{3\sqrt{2}}$. The area over which $-dS_2/ds_\mu$ may be neglected is approximately that comprised between $\mu^2 = \frac{1}{3}$ and $\mu^2 = \frac{1}{8}$.

A graph, fig. 2, of F is shown. In this figure, OA is the radius of the disc. The

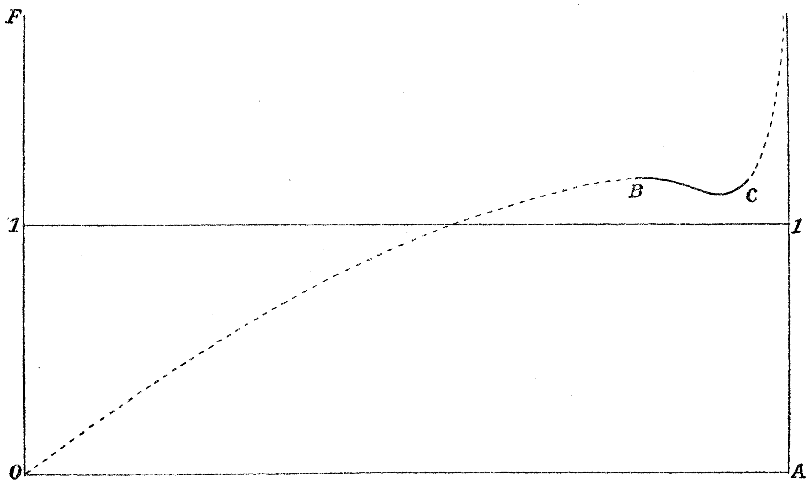

Fig. 2.

line 1—1 represents unit ratio. The continuous portion BC of the curve represents the ratio of the amplitudes in the actual motion with tolerable accuracy. The complete curves for the amplitude and phase can be obtained without difficulty; this can be left to those readers who are interested.

The formulæ should not be used for values of κ greater than $3\cdot25$, but up to this value the errors are tolerably small.

When κ lies in the neighbourhood of 3 and ρ is large the expression for the scattered wave is

$$S = \frac{e^{-i(\rho-\frac{1}{2}\pi)}}{\rho}\left[3C_1E_1 - i\frac{14}{\pi}D_3E_3\right],$$

$$= \frac{e^{-i(\rho-\frac{1}{2}\pi)}}{\rho}\left[3\left(1+\frac{\kappa^2}{25}\right)E_1 - i\frac{14}{\pi}\cdot\frac{1}{50}\left(\frac{\kappa}{3}\right)^5 E_3\right].$$

The contrast of this expression with that in which κ is small is very conspicuous.

The Circular Aperture.

When a solution of such a problem as the transmission of waves through a circular aperture is obtained under the restriction that κ is small, the result is very vague. There is no means, by RAYLEIGH's method, of obtaining an upper limit to the value of κ, for which the solution will be valid with reasonable accuracy.

For the consideration of the circular aperture we enter upon a different type of problem. It has already been stated that, for certain applications, a second solution of equation (4) is required. This solution, which we denote by E'_n, is analogous to that which has been developed as the second solution of equation (5) and denoted by H'_n.

The solution E'_n may be expressed in terms of E_n by the equation

$$E'_n = E_n \,.\, \log \frac{1-\mu}{1+\mu} + G_n, \qquad (114)$$

where, if n be even, G_n can be developed in a series of ascending odd powers of μ, and *vice versa*.

The solutions E_n and E'_n are both required when the problem of the reflection of waves from a hyperboloid of revolution is under consideration.

Now this solution E'_n has a logarithmic infinity when $\mu = 1$, and this requires explanation.

Let us consider an elementary wave propagated, after reflection, between the two right circular zones $A_1C_1B_1$ and $A_2C_2B_2$, fig. 3. On account of symmetry we must

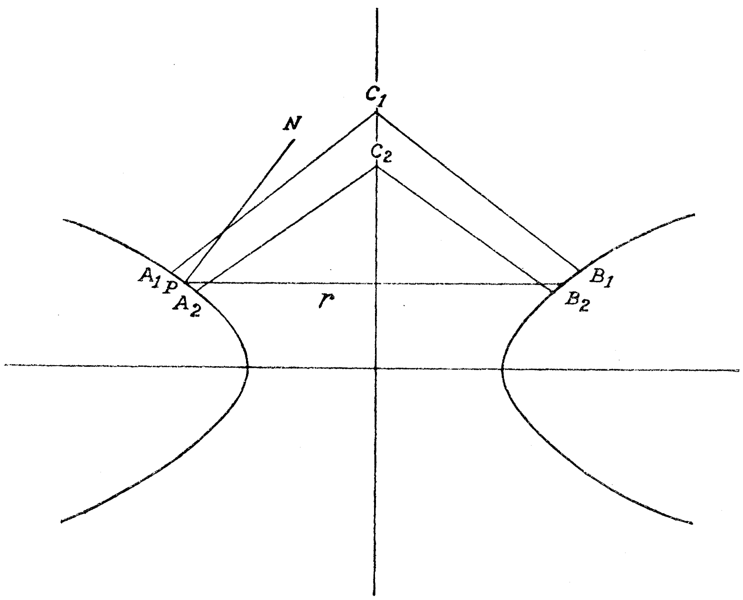

Fig. 3.

assume that all waves incident symmetrically will, after reflection, pass through the common axis of revolution.

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Consider the elementary ring of the surface of one of the family of hyperboloids comprised between the two zones. The area of the ring is $2\pi r \delta s_\beta$. Let the normal PN to the surface make an angle ψ with the generating line of one of the zones.

If E be the energy propagated through unit area normal to its direction of propagation, then the energy passing through the ring is

$$E \,.\, 2\pi r \delta s_\beta \,.\, \cos \psi.$$

When $r = 0$, that is when $\mu = 1$, E must be infinite, and thus an infinity must enter into the solution which represents the reflected waves.

The most general elementary solution of the wave equation is

$$\phi_n = (H_n + AH'_n)(E_n + BE'_n),$$

where A and B are arbitrary constants. If a boundary consists of one of the hyperboloids, B is determined by the condition to be satisfied at the boundary. If this hyperboloid is the plane screen in which there is a circular aperture, it is represented algebraically by the equation $\mu = 0$.

If the boundary condition is expressed by putting $\phi_n = 0$ when $\mu = 0$, then, since E_n and E'_n do not vanish simultaneously when $\mu = 0$, B must vanish for all odd values of n.

Similarly, if the boundary condition is expressed by putting $d\phi_n/d\mu = 0$ when $\mu = 0$, B must vanish for all even values of n.

The circular aperture in a plane screen is therefore of special simplicity, and can be treated analytically without great labour.

Plane Waves of Sound Incident Normally upon an Infinitely Thin Plane Screen in which there is a Circular Aperture.

Using our current co-ordinates, and omitting the time factor $e^{i\kappa ct}$, the plane wave is denoted by

$$I = e^{i\kappa z},$$

so that z diminishes as t increases. The reflected wave, in the absence of the circular aperture, is denoted by

$$R = e^{-i\kappa z}.$$

These two solutions of the wave equation satisfy the necessary condition over the surface of the plane screen on the incident side. Over the plane of the aperture they will make

$$I + R = 2$$

and

$$\frac{d}{\kappa dz}(I + R) = \frac{d}{\mu d\rho}(I + R) = 0.$$

In addition to these two waves there will therefore exist on the incident side what may be called a scattered wave.

In the incident space, which will be designated by the suffix 1, let the scattered wave be S_1.

In the transmitted space, which will be designated by the suffix 2, let the transmitted wave be S_2.

Both S_1 and S_2 must be divergent waves. With our previous notation let

$$S_2 = a\,(a_0\psi_0 E_0 + a_2\psi_2 E_2 + \ldots),$$
$$S_1 = b\,(b_0\psi_0 E_0 + b_2\psi_2 E_2 + \ldots).$$

These two expressions satisfy the boundary conditions over the surface of the screen. Over the plane of the aperture we must have

$$\frac{d}{d\rho}(S_1 + S_2) = 0. \quad \ldots \quad (115)$$

This equation makes the velocity continuous over the aperture, since S_1 is propagated into space 1 and S_2 into space 2, ρ changing sign as the plane of the aperture is crossed.

The condition (115) is fulfilled if

$$b = -a$$
$$b_0 = a_0$$
$$b_2 = a_2, \quad \text{etc.}$$

Again over the plane of the aperture the equation

$$S_2 = 2 + S_1 \quad \ldots \quad (116)$$

must be identically satisfied. Hence over the aperture the equation

$$\tfrac{1}{2}(S_2 - S_1) = a\,(a_0\psi_0 E_0 + a_2\psi_2 E_2 + \ldots) = 1 \quad \ldots \quad (117)$$

must be identically satisfied.

Since unity can be expended in a series of the functions E_n, where n is always even, only the even solutions of the wave equation are required. Now substitute for unity in equation (117) its expansion given by (18), viz.,

$$2\left(\frac{E_0}{e_0} + \frac{\beta_{20}}{e_2}E_2 + \frac{\beta_{40}}{e_4}E_4 + \ldots\right),$$

then, over the aperture where $\rho = 0$, we must have, if $a = (\tfrac{1}{2}\pi)^{\frac{1}{2}}$,

$$a_0 \cdot \tfrac{1}{2}e_0\,(1 + \tfrac{1}{2}\pi A_0) = 1$$
$$a_2 \cdot \tfrac{1}{2}e_2\,(1 + \tfrac{1}{2}\pi A_2) = -1$$
$$a_4 \cdot \tfrac{1}{2}e_4\,(1 + \tfrac{1}{2}\pi A_4) = 1, \quad \text{etc.,}$$

where

$$A_n = -iC_n/D_n.$$

These follow from equations (97), and from the relations between the β's and the γ's When ρ is large

$$\psi_n = C_n \frac{1}{\rho}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} e^{-i(\rho-\frac{1}{2}\pi n-\frac{1}{2}\pi)}.$$

Hence, when ρ is large,

$$S_2 = \frac{1}{\rho} e^{-i\rho+i\frac{1}{2}\pi}\{a_0C_0E_0 + a_2C_2E_2 + a_4C_4E_4 + \ldots\},$$

where

$$a_nC_n = \frac{2}{e_n}\cdot\frac{C_nD_n}{D_n - i\frac{1}{2}\pi C_n}. \qquad (118)$$

For values of κ not exceeding 3, D_n diminishes rapidly as n increases. When $n = 0$ the expression for D_0 is somewhat complicated. If the common denominator of (87) and (88) be denoted by Δ, then

$$\Delta = A_{n+1}\, A_{n+3} - L_{n+1}\, N_{n+3}.$$

Also, with tolerable accuracy,

$$\Delta \,.\, D_0 = -B_{n,\,n+1}\,(A_{n+3} + L_{n+1}) + B_{n,\,n+3}\,(N_{n+3} + A_{n+1}),$$

the omitted terms being insignificant.

Now

$$A_{n+3} = -12\left(1 - \frac{2}{9\,.\,5\,.\,3}\kappa^2 - \frac{1}{6\,.\,5\,.\,3}\kappa^2\gamma_{02}\right), \qquad (119)$$

where γ_{02} is given by (93).

By a simple calculation it is easy to show that the factor within brackets upon the right-hand side of (119) differs little from unity for the range of values of κ lying between 0 and 3. Hence we may put

$$A_{n+3} = -12.$$

Under these circumstances Δ can be obtained in terms of γ_{02} by the equation

$$\Delta = 24\left\{1 - \frac{2}{5\,.\,3}\kappa^2 - \frac{1}{7\,.\,5\,.\,5\,.\,2}\kappa^4 - \frac{1}{5\,.\,3}\kappa^2\gamma_{02}\right\}.$$

Again

$$\Delta\,.\,D_0 = B_{n,\,n+1}\,(12 + \tfrac{2}{5}\kappa^2) - B_{n,\,n+3}\left(2 - \frac{2}{7\,.\,3}\kappa^2 - \frac{2}{5\,.\,3}\kappa^2\gamma_{02}\right),$$

where $B_{n,\,n+1}$ and $B_{n,\,n+3}$ are given by (64) in terms of γ_{02} and γ_{04}.

These equations are sufficient for the evaluation of D_0, and, as in the case of D_1, they show that D_0 increases very rapidly as κ increases.

The values of D_n decrease with great rapidity as n increases, so that for the range of values of κ from 0 to 3 only D_2 and D_4 need be retained. These are obtained at once from (99), and the values of C_n and e_n can be written down by means of (96) and (17) respectively. If κ be not greater than 2

$$S_2 = \frac{i}{\rho} e^{-i\rho} a_0 C_0 E_0$$

with tolerable accuracy. This expression shows a marked variation in amplitude for different values of θ, in contrast to the expression obtained as a first approximation when κ is small.

When κ is small

$$a_0 C_0 = D_0/(-i\tfrac{1}{2}\pi)$$
$$= \frac{2\kappa}{i\pi}.$$

Hence, since $\rho = \kappa_1 R$ when ρ is large,

$$S_2 = \frac{2a}{\pi} \cdot \frac{1}{R} e^{-i\kappa_1 R}.$$

This is in agreement with RAYLEIGH's formula (see LAMB's "Hydrodynamics," p. 513, 4th Ed.), since

$$S_2 = -S_1.$$

When $\kappa > 3$ the evaluation of the coefficients is very tedious. We shall limit our investigation, when $\kappa > 3$, to a consideration of the first term in the expression for S_2. So far as diffraction phenomena are concerned the most remarkable results lie in the variations of E_0 as κ is increased up to 5.

Now in equation (55) we have found an expression for E_0 which is valid with tolerable accuracy even when κ is as great as 5. This expression may be written in the form

$$E_0 = P_0 + \frac{3 \cdot 5}{2\kappa^2} b_0 \left\{ P_2 + \frac{3\kappa^2}{5 \cdot 7 \cdot 5} P_4 + \frac{\kappa^4}{7 \cdot 5 \cdot 11 \cdot 3 \cdot 7} P_6 \right\} \cdot \cdot \cdot \cdot \quad (120)$$

Here

$$b_0 = -3\alpha + (9\alpha^2 + 6A)^{\frac{1}{2}},$$

where

$$\alpha = 1 - \frac{2\kappa^2}{7 \cdot 9} - \frac{9 \cdot 16\kappa^4}{5 \cdot 7 \cdot 7 \cdot 9 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$$

and

$$A = \frac{4\kappa^4}{2 \cdot 27 \cdot 5}.$$

The maximum value of (120) clearly occurs when $\mu = 1$, and little increase in accuracy is to be attained by retaining the term containing P_6 unless perhaps when μ is small. We shall therefore retain only that term in P_6 which is independent of μ. Now

$$P_2 = \tfrac{3}{2}\mu^2 - \tfrac{1}{2}$$

$$P_4 = \frac{5.7}{2.4}\mu^4 - \frac{5.3}{2.2}\mu^2 + \frac{1.3}{2.4},$$

and the term independent of μ in P_6 is

$$= \frac{1.3.5}{2.4.6}.$$

Hence, upon substitution and rearrangement, (120) becomes

$$\frac{E_0}{b_0} = (\tfrac{3}{4}\mu^2)^2 + \left(\frac{3.5}{\kappa^2} - \frac{3.3}{2.7}\right)(\tfrac{3}{4}\mu^2) + \frac{1}{b_0} - \tfrac{1}{4}\left(\frac{3.5}{\kappa^2} - \frac{3.3.3}{4.7.5} + \frac{5\kappa^2}{7.11.7.8}\right). \quad (121)$$

It is remarkable that, when κ lies close to the value 5, both the coefficient of μ^2 and the term independent of μ in (121) very nearly vanish. In this case, therefore, E_0 varies very nearly as μ^4, in marked contrast to its value when κ is small, viz., unity. This fact has an interesting application later.

The Oscillations of a Gas within a Rigid Ellipsoidal Envelope.

The most interesting elementary solutions of the wave equation are those denoted by

$$\phi = H_n E_n$$

and

$$\phi = H'_n E_n.$$

These functions represent the possible modes of motion of a gas within an ellipsoidal envelope.

If β_0 denote the envelope the boundary condition is satisfied if $\delta\phi/\delta\rho = 0$ when $\rho = \kappa\beta_0$. The roots of the equation $\delta\phi/\delta\rho = 0$ determine the normal modes.

There are three cases of importance which we shall consider in some detail. These are

$$\text{I.} \quad \phi = H_0 E_0$$

$$\text{II.} \quad \phi = H'_0 E_0$$

$$\text{III.} \quad \phi = H_1 E_1.$$

I. In this case

$$H_0 = \rho^{-\frac{1}{2}}(F_0 + \gamma_{02}F_2),$$

where

$$\rho^{-\frac{1}{2}}F_0 = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{\sin\rho}{\rho}$$

and

$$\rho^{-\frac{1}{2}} F_2 = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left\{\left(\frac{3}{\rho^3} - \frac{1}{\rho}\right) \sin \rho - \frac{3}{\rho^2} \cos \rho\right\}.$$

The two first terms in the expansion of H_0 are sufficient if κ be limited to values not much greater than unity. If κ be less than $\frac{1}{2}$ it is sufficient to write $H_0 = \rho^{-\frac{1}{2}} F_0$. The normal modes are then given by the equation

$$\tan \rho = \rho,$$

of which the gravest mode corresponds to

$$\frac{\rho}{\pi} = 1 \cdot 4303.$$

This is, of course, the same as in the radial oscillations within a sphere, the solution of which is well known. When κ becomes large the equation to determine the normal modes becomes very cumbersome, but it may be noted that when κ lies in the neighbourhood of 5 E_0 varies very nearly as μ^4. Hence the vibrations are sensible only within a region close to the axis of symmetry where μ is not very different from unity.

The diametral plane $\mu = 0$, $\beta = 0$, is a node for the solution $H_0 E_0$.

II. In this case μ is everywhere positive and ρ changes sign upon crossing the surface $\rho = 0$.

$$H'_0 = \rho^{-\frac{1}{2}} (F_0 + \gamma_{02} F_2) \cot^{-1} \frac{\rho}{\kappa} + \rho^{-\frac{1}{2}} \varepsilon_{01} F_1,$$

where

$$\rho^{-\frac{1}{2}} F_1 = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left\{\frac{\sin \rho}{\rho^2} - \frac{\cos \rho}{\rho}\right\}.$$

The above expression for H'_0 is sufficient if κ be not much greater than unity. If κ be less than $\frac{1}{2}$, γ_{02} may be neglected and ε_{01} takes its first approximation, viz., $-\kappa$. Further in the determination of all the normal modes $\cot^{-1} \frac{\rho}{\kappa}$ may, with ample accuracy, be replaced by κ/ρ. Hence

$$H'_0 = \rho^{-\frac{1}{2}} \left(\frac{\kappa}{\rho} F_0 - \kappa F_1\right)$$

$$= \kappa \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{\cos \rho}{\rho}.$$

The normal modes are then given by

$$\rho \tan \rho + 1 = 0,$$

of which the gravest mode corresponds to

$$\frac{\rho}{\pi} = 1 - \frac{1}{\pi^2 - 1} \quad \text{approximately.}$$

This root lies between that of Case I and that of Case III. The motion is one in which the gas oscillates backwards and forwards through an aperture in the rigid partition $\mu = 0$.

In this case the same remark applies when κ lies in the neighbourhood of 5 as in the previous case.

III. In this case

$$H_1 = \rho^{-\frac{1}{2}} F_1,$$

which is sufficient even if κ be as great as unity. Hence the normal modes are given by

$$\frac{d}{d\rho}\left\{\frac{\sin \rho}{\rho^2} - \frac{\cos \rho}{\rho}\right\} = 0,$$

or

$$\tan \rho = \frac{2\rho}{2 - \rho^2},$$

of which the gravest mode corresponds to

$$\frac{\rho}{\pi} = 0 \cdot 6625.$$

In this mode the gas sways backwards and forwards across the diametral plane $\mu = 0$, $\beta = 0$.

This case reduces therefore to the type of vibrations within a sphere which RAYLEIGH calls diametral.

By the introduction of the function E'_n, other types of vibration could be considered.

Summary.

Upon the Ellipsoidal Functions depends the solution of the scattering of waves by a thin circular disc, and also of the passage of waves through a circular aperture in a thin screen.

An attempt is made in this paper to arrive at some results of importance. The expansion of an arbitrary function in a series of Ellipsoidal Functions is obtained. An expression for the solution convergent at all points and suitable to represent a divergent wave is also obtained.

RAYLEIGH'S solution for the scattering of plane waves, when the ratio of the diameter of the disc to the wave-length is small, is shown to be slightly in error.

In order to make the theory applicable to problems of importance, considerable attention has been paid to the coefficients. It is shown that for values of the above ratio somewhat greater than unity the expansion of the coefficients in ascending powers of the ratio is inadmissible, and a suitable expansion is obtained.

Among the problems, briefly treated, of the oscillations of a gas within a spheroidal envelope is one in which the diametral plane of the spheroid is occupied by a thin rigid partition pierced by a circular hole. In certain cases, when the above ratio is rather greater than unity, the oscillations are sensible only in the neighbourhood of the axis of symmetry.

The above problems appear hitherto to have been considered intractable, but so far as the author is aware the chief difficulty has lain in obtaining a suitable expression for the divergent wave and in dealing with the coefficients. It is not to be expected that optical problems in which the above ratio is large can be solved by these methods, but a treatment of the theory when the ratio is neither large nor small has been much needed.

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